Australian
Category
Seminar

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The fundamental theorem of n-quasi-category theory

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<u>Theorem</u>. A morphism of n-quasi-categories is an equivalence iff it is essentially surjective on objects and fully faithful.

§1. <u>Rezk's construction</u>.

<u>Def</u>. A <u>presentation</u> is a pair (C, S) consisting of a small category C and a small set S of morphisms in $sPsh(C) = [C^{op}, sSet]$.

A presentation determines a model category: the Bousfield localisation of the injective model structure on $sPsh(C)$ at the set S.

Given a presentation (C, S), Rezk constructs a presentation $(\Theta C, Se_C \cup Cpt_C \cup V[1](S))$.

- ΘC is the wreath product $\Delta \wr C$ (Berger):
  objects: $[m](c_1, \ldots, c_m)$, $m \geq 0$, $c_i \in C$.
  morphisms: $[m](c_1, \ldots, c_m) \xrightarrow[(\delta, \{f_{ij}\})]{} [n](d_1, \ldots, d_n)$

$0 \xrightarrow{c_1} 1 \cdots m-1 \xrightarrow{c_m} m$
"free category on enriched graph"

$[m] \xrightarrow{\delta} [n]$ in Δ,

$f_{ij} : c_i \longrightarrow d_j$ $\quad 1 \leq i \leq m$, $\delta(i-1) < j \leq \delta(i)$

$\left(" \; c_i \xrightarrow{(f_{ij})} d_{\delta(i-1)+1} \times \cdots \times d_{\delta(i)} \; " \right)$.

Joyal's categories Θ_n are defined inductively:
$\Theta_0 = 1$, $\Theta_{n+1} = \Theta\Theta_n$. $\quad (\Theta_1 = \Delta)$.

- Se_C is the set of "Segal maps" in $sPsh(\Theta C)$:
  each $[m](c_1, \ldots, c_m) \in \Theta C$, get
  $se^{(c_1, \ldots, c_m)} : G[m](c_1, \ldots, c_m) \longrightarrow F[m](c_1, \ldots, c_m)$

where $\Theta_m C \xrightarrow{F} sPsh(\Theta_m C)$ is the Yoneda emb.
&

$$G[m](c_1,\dots,c_m) = F[1](c_1) \underset{F[0]}{+} \cdots \underset{F[0]}{+} F[1](c_m).$$

"horizontal spine".

$$sPsh(\Theta_m C) \underset{T^*}{\overset{T_\#}{\underset{\perp}{\rightleftarrows}}} sPsh(\Delta)$$

is the simplicially enriched adjunction induced by the functor

$$T : \Delta \longrightarrow sPsh(\Theta_m C)$$
$$[n] \longmapsto \big([m](c_1,\dots,c_m) \mapsto \Delta([m],[n])\big)$$

(if C has a terminal object t, then T is

$$\Delta \longrightarrow \Theta_m C \longrightarrow sPsh(\Theta_m C)$$
$$[n] \longmapsto [n](t,\dots,t) \longmapsto F[n](t,\dots,t)$$

).

By Yoneda, if $X \in sPsh(\Delta)$, then

$$(T_\# X)[m](c_1,\dots,c_m) \cong \int^{[n]\in\Delta} X_n \times \Delta([m],[n])$$

$$\cong X_m .$$

T^* sends an object of $sPsh(\Theta_m C)$ to its <u>underlying simplicial space</u>:

$$(T^* X)_m = \underline{\mathrm{Hom}}(T[m], X) \quad \Big(= X[m](t,\dots,t) \Big)$$

(at least if C has a terminal object.) $T_\#$ is ff.
[C connected should do.]

~~This adjunction is Quillen wrt injective model structures~~

- $$(F[0]+F[0])/sPsh(\Theta_{st}C) \underset{\substack{\text{~~Hom~~} \\ M}}{\overset{V[1]}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} sPsh(C)$$

"suspension-hom adjunction"

is the simplicially enriched adjunction determined by the functor

$$C \longrightarrow (F[0]+F[0])/sPsh(\Theta_{st}C)$$

$$c \longmapsto \begin{array}{c} F[0]+F[0] \\ \downarrow \\ F[1](c) \end{array} \qquad \bullet \xrightarrow{c} \bullet$$

The left adjoint sends $A \in sPsh(C)$ to $V[1](A) \in sPsh(\Theta_{st}C)$

$$[m](c_1,\ldots,c_m) \longmapsto 1 + \sum_{i=1}^{m} A(c_i) + 1 .$$

The right adjoint sends $X \in sPsh(\Theta_{st}C)$, $x, y \in X_{[0]}$ to the ~~pullback~~ object $M_X(x,y) \in sPsh(C)$:

$$\begin{array}{ccc} M_X(x,y)_c & \longrightarrow & X[1](c) \\ \Big\downarrow & \lrcorner & \Big\downarrow \\ 1 & \longrightarrow & X_0 \times X_0 \end{array} \qquad \text{pb in sSet,}$$

~~This adjunction is Quillen wrt injective model str.~~

- Cpt_C consists of the single map
$$T_\#(E) \longrightarrow T_\#(1) \qquad \left(E \in sPsh(\Delta) \text{ is discrete nerve of } \cdot \overset{\simeq}{\rightleftarrows} \cdot\right)$$

- $V[1](S)$ consists of the morphisms
$$V[1](A) \xrightarrow[V[1](f)]{} V[1](B) \quad \text{in } sPsh(\Theta_{st}C)$$

for $A \xrightarrow{f} B$ in C.

~~A fibrant object~~ in
The presentation $(\Theta_n C, \mathrm{Seg}_C \cup \mathrm{Cpt}_C \cup V[1](S))$ determines a model structure on $sPsh(\Theta_n C)$, whose fibrant objects are called Θ_n-spaces over (C,S).
(I'll call them Rezk objects.)

~~With these model st~~

Ex. $C=1$, $S=\phi$, we get the model structure for complete Segal spaces on $sPsh(\Delta)$.

The simplicial adjunctions

$$sPsh(\Theta_n C) \underset{T^*}{\overset{T_\#}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} sPsh(\Delta)$$

$$(F[0] \amalg F[0]) / sPsh(\Theta_n C) \underset{M}{\overset{V[1]}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} sPsh(C)$$

are ~~simplicial~~ Quillen adjunctions wrt these model structures.

Def. • A morphism $X \xrightarrow{f} Y$ of complete Segal spaces (in $sPsh(\Delta)$) is essentially surjective on objects (eso) if the functor $hoX \xrightarrow{ho(f)} hoY$ between homotopy categories is eso.

$$\left(Cat \underset{N}{\overset{ho}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} sPsh(\Delta), \quad \begin{array}{ccc} \Delta \times \Delta & \longrightarrow & Cat \\ ([m],[n]) & \longmapsto & [m] \times \widetilde{[n]} \end{array} \right)$$

• A morphism $X \xrightarrow{f} Y$ of Rezk objects in $sPsh(\Theta_n C)$ is eso if $T^* X \xrightarrow{T^*(f)} T^* Y$ is eso.

Def. A morphism $X \xrightarrow{f} Y$ of Rezk objects in $sPsh(\Theta_n C)$ is fully faithful (ff) if $M_X(x,y) \xrightarrow{f} M_Y(fx,fy)$ is an equivalence in $sPsh(C)$.

<u>Theorem.</u> A morphism of Rezk objects in $\mathrm{sPsh}(\textcircled{wr}C)$ is an equivalence iff it is eso and ff.

<u>Proof:</u> By construction, a morphism $X \xrightarrow{f} Y$ of Rezk objects is an equivalence iff

$$X_{m(c_1,\ldots,c_m)} \xrightarrow{f} Y_{m(c_1,\ldots,c_m)}$$

is an equivalence of Kan complexes $\forall [m](c_1,\ldots,c_m) \in \textcircled{wr}C$.
By the Segal property, it suffices to check this for $[0]$ and $[1](c)$:

$$\begin{tikzcd}
X_{m(c_1,\ldots,c_m)} \arrow[r, "f"] \arrow[d, "\sim"'] & Y_{m(c_1,\ldots,c_m)} \arrow[d, "\sim"] \\
X_{1(c_1)} \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_{1(c_m)} \arrow[r, "f \underset{f}{\times} \cdots \underset{f}{\times} f"'] & Y_{1(c_1)} \underset{Y_0}{\times} \cdots \underset{Y_0}{\times} Y_{1(c_m)}
\end{tikzcd}$$

Now, $T^*(f) : T^*X \longrightarrow T^*Y$ is an eso & ff map of complete Segal spaces:
since $M_{T^*X}(x,y) \cong \Gamma(M_X(x,y))$

$$\left(\begin{tikzcd}
(F[0]+F[0])/\mathrm{sPsh}(\textcircled{wr}C) \arrow[r, shift right, "M"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "T^*"'] \arrow[d, phantom, "\dashv"] & \mathrm{sPsh}(C) \arrow[l, shift right, "V[1]"'] \arrow[d, shift right, "\Gamma"'] \arrow[d, phantom, "\dashv"] \\
(F[0]+F[0])/\mathrm{sPsh}(\Delta) \arrow[u, shift right, "T_\#"'] \arrow[r, shift right, "M"'] \arrow[r, phantom, "\perp"] & \mathrm{sSet} \arrow[l, shift right, "V[1]"'] \arrow[u, shift right, "\Delta"']
\end{tikzcd}\right)$$

$$\begin{tikzcd}
F[0]+F[0] \arrow[d] & \\
1 \sqcup \Delta[1] & \Delta[0] \arrow[lu, mapsto]
\end{tikzcd}$$

Hence, by ~~a~~ an ~~result~~ argument of Rezk using the completeness condition, $X_{[0]} \xrightarrow{\sim} Y_{[0]}$

is a htpy equivalence.

iff $\Leftrightarrow$ $\cancel{X_{[1](c_1,\dots,c_m)} \xrightarrow{f} Y_{[1]}}$

$$\begin{tikzcd} X_{[1](c)} \arrow[r, "f_{1(c)}"] \arrow[d] & Y_{[1](c)} \arrow[d] \\ X_{[0]} \times X_{[0]} \arrow[r, "f_0 \times f_0"'] & Y_{[0]} \times Y_{[0]} \end{tikzcd}$$

is a homotopy pullback $\forall c \in C$.

Hence $f_{1(c)} : X_{[1](c)} \longrightarrow Y_{[1](c)}$ is a

homotopy equiv. $\square$

Starting with the presentation $(\mathbb{1}, \emptyset)$, Rezk's construction inductively defines a model structure on $sPSh(\Theta_n)$ for each $n \geq 0$. The fibrant objects of $sPSh(\Theta_n)$ are called

Rezk Θ_n-spaces, and are a model for (∞, n)-categories.

(special case)

Cor. A morphism of Rezk Θ_n-spaces is an equivalence iff it is eso & ff (= equivalence on hom Θ_{n-1}-spaces).

§2. Equivalences of n-quasi-categories.

For each $n \geq 1$, Ara defined a model structure on $[\Theta_n^{op}, Set]$ whose fibrant objects are called n-quasi-categories; in the case n=1 this is Joyal's model structure for quasi-categories [cf. ?]

Generalizing results of Joyal & Tierney, Ara defined two Quillen equivalences:

$$\underbrace{[\Theta_n^{op}, Set] \underset{t^!}{\overset{t_!}{\rightleftarrows}} [\Theta_n^{op}, sSet]}_{(1)}, \quad \underbrace{[\Theta_n^{op}, sSet] \underset{i^*}{\overset{p^*}{\leftrightarrows}} [\Theta_n^{op}, Set]}_{(2)}$$

where (1) is induced by the functor

$$\Theta_n \times \Delta \longrightarrow [\Theta_n^{op}, Set]$$
$$(T, [n]) \longmapsto \Theta[T] \times \widetilde{\Delta[n]}$$

(2) is ~~induced by the~~ defined by precomposition with the adjunction

$$\Theta_n \times \Delta \underset{i}{\overset{p}{\rightleftarrows}} \Theta_n$$

where $p(T, [n]) = T$

$i(T) = (T, [0])$.

- $[\Theta_n^{op}, Set] \underset{\tau^*}{\overset{\pi^*}{\leftrightarrows}} [\Delta^{op}, Set]$

defined by precomposition with $\Theta_n \underset{\tau}{\overset{\pi}{\rightleftarrows}} \Delta$

$\pi([m](c_1, \dots, c_m)) = [m]$

$\tau[m] = [m](t, \dots, t)$

(this is the unenriched version of the $T_\# \dashv T^*$ simplicial adjunction)

This is a Quillen adjunction:
right adjoint sends n-quasi-cat to its underlying quasi-cat.

~~$n?\ (n+1)?\ [\Theta_n^{op}, Set] \longrightarrow$~~

• $\underline{n=1.}$

$$\begin{tikzcd} (1+1)/[\Delta^{op}, Set] \arrow[r, shift right, "Hom"'] \arrow[r, phantom, "\perp"] & {[\Delta^{op}, Set]} \arrow[l, shift right, "\Sigma"'] \end{tikzcd}$$

$$\Delta \longrightarrow (1+1)/[\Delta^{op}, Set]$$

$$\begin{tikzcd} {[m]} \arrow[r, mapsto] & \partial\Delta[1]\times\Delta[m] \arrow[r] \arrow[d] & \partial\Delta[1] \arrow[d] \\ & \Delta[1]\times\Delta[m] \arrow[r] & \Sigma(\Delta[m]) \end{tikzcd}$$

right adjoint sends X, $x,y\in X$ to the pullback

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\Delta[1]} \arrow[d] \\ 1 \arrow[r, "{(x,y)}"'] & X\times X \end{tikzcd}$$

pb in sSet

This is a Quillen adjunction between Joyal & Kan-Quillen model structures

• $\underline{n\geq 2.}$

$$\begin{tikzcd} (1+1)/[\Theta_n^{op}, Set] \arrow[r, shift right, "Hom"'] \arrow[r, phantom, "\perp"] & {[\Theta_{n-1}^{op}, Set]} \arrow[l, shift right, "\Sigma"'] \end{tikzcd}$$

~~left~~ $\Theta_{n-1} \longrightarrow (1+1)/[\Theta_n^{op}, Set]$

$$\begin{tikzcd} T \arrow[r, mapsto] & 1+1 \arrow[d] \\ & {[1](T)} \end{tikzcd}$$

right adjoint sends X, $x,y\in X$ to $X(x,y)\in[\Theta_{n-1}^{op}, Set]$

$$\begin{tikzcd} X(x,y)_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{1(T)} \arrow[d] \\ 1 \arrow[r, "{(x,y)}"'] & X_0\times X_0 \end{tikzcd}$$

pb in Set

Def. A morphism $X \xrightarrow{f} Y$ of n-quasi-categories is:

- **eso** if $\tau^*X \xrightarrow{\tau^*(f)} \tau^*Y$ is an eso morphism of quasi-categories (= eso on homotopy cats)
- **ff** if $X(x,y) \xrightarrow{f} Y(fx,fy)$ is an equivalence of Kan complexes / (n−1)-quasi-cats
  $n=1$ / $n \geq 1$.

~~Theorem. A morphism of n-quasi-categories is an equivalence iff it is eso & ff.~~

~~proof. By the Quillen equivalence $t_! \dashv t^!$, $X \xrightarrow{f} Y$ is an equivalence of n-quasi-cats iff $t^!X \xrightarrow{t^!f} t^!Y$ is an~~

Prop. Let $X \xrightarrow{f} Y$ be a morphism of n-quasi-categories. Then f is eso / ff iff $t^!(f) : t^!X \to t^!Y$ is an eso / ff morphism of Rezk Θ_n-spaces.

proof. eso: $X \xrightarrow{f} Y$ ~~n-quasi-cat~~ eso iff $\tau^*X \xrightarrow{\tau^* f} \tau^*Y$ eso morphism of quasi-cats

iff $ho(\tau^*X) \xrightarrow{ho(\tau^*f)} ho(\tau^*Y)$ eso.

Now, $t^!\tau^* \cong \tau^* t^!$,

$$\begin{tikzcd}
{[\Theta_n^{op}, Set]} \arrow[r, shift right, "t^!"'] \arrow[d, shift right, "\tau^*"'] & {[\Theta_n^{op}, sSet]} \arrow[l, shift right, "t_!"'] \arrow[d, shift right, "\tau^*"'] \\
{[\Delta^{op}, Set]} \arrow[r, shift right, "t^!"'] \arrow[u, shift right, "i^*"'] & {[\Delta^{op}, sSet]} \arrow[l, shift right, "t_!"'] \arrow[u, shift right, "\#"']
\end{tikzcd}
\qquad
\begin{array}{c}
\Delta[k] \times \tilde{\Delta}[m] \\
\uparrow \\
\Delta[k] \square \Delta[m]
\end{array}$$

$t^! X \xrightarrow{t^!(f)} t^! Y$ eso iff $T^*(t^! X) \xrightarrow{T^* t^!(f)} T^*(t^! Y)$ eso

iff ho of it is eso.

so follows from:

if $X \xrightarrow{f} Y$ morphism of quasi-cats,
then f eso iff $t^!(f)$ eso morphism of CSS's

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, shift right, "N"'] \arrow[r, phantom, "\perp"] & {[\Delta^{op}, Set]} \arrow[l, shift right, "\mathrm{ho}"'] \arrow[r, shift right, "t^!"'] \arrow[r, phantom, "\perp"] & {[\Delta^{op}, sSet]} \arrow[l, shift right, "t_!"'] \end{tikzcd}$$

this is because $\mathrm{ho}(t^! X) \cong \mathrm{ho}(X)$
(non-trivial)

since $\mathrm{ho}(t^! X) \cong \mathrm{ho}(t_! t^! X)$

& $t_! t^! X \xrightarrow{\sim} X$
is a bij-on-objs weak categorical equivalence.

Pf: ~~*~~

$\underline{n=1}$. & $M_{t^! X}(x,y) = k^!(X(x,y))$.

$k^! X(x,y) \xrightarrow{\sim} X(x,y)$
triv fib since $X(x,y)$ Kan

$\underline{n \geq 2}$ use $i^*(M_{t^! X}(x,y)) \cong X(x,y)$.

$$\begin{tikzcd} (1+1)/[\Theta_n^{op}, sSet] \arrow[r, shift right, "M"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "i^*"'] \arrow[d, phantom, "\vdash"] & {[\Theta_{n-1}^{op}, sSet]} \arrow[l, shift right, "V[1]"'] \arrow[d, shift right, "i^*"'] \arrow[d, phantom, "\vdash"] \\ (1+1)/[\Theta_n^{op}, Set] \arrow[u, shift right, "p^*"'] \arrow[r, shift right, "\mathrm{Hom}"'] \arrow[r, phantom, "\perp"] & {[\Theta_{n-1}^{op}, Set]} \arrow[l, shift right, "\Sigma"'] \arrow[u, shift right, "p^*"'] \end{tikzcd}$$

$$\begin{array}{c} 1+1 \\ \downarrow \\ F[1](T) \\ \uparrow \\ T \end{array}$$

i^* preserves & reflects weak equivalences btwn fibrants □

<u>Theorem</u>. A morphism of n-quasi-categories is an equivalence iff it is eso & ff.

<u>Proof</u>. $X \xrightarrow{f} Y$ is an equivalence of n-quasi-cats iff $t^! X \xrightarrow{t^! f} t^! Y$ is an equivalence of Rezk Θ_n-spaces (since $t_! \dashv t^!$ is a Quillen equivalence).

But $t^! f$ is an equiv iff its eso & ff — but we just saw that this holds precisely when f is eso & ff □

~~Non-inductive characterisation:~~

- ~~for each $0 \le k \le n$, say a morphism $X \xrightarrow{f} Y$ of n-quasi-categories is k-surjective if for every parallel pair k-cell $D_k \xrightarrow{y} Y$ in Y there exists a k-cell $D_k \xrightarrow{x} X$ & an invertible $(k+1)$-cell $fx \cong y$ in Y.~~

~~(ie an invertible morphism in the $(n-k-1)$-quasi-category $\mathrm{Hom}_Y(fx, y)$)~~

- ~~$X \xrightarrow{f} Y$ is fully faithful on n-cells if~~

for $\forall$ x y in X parallel $(k-1)$-cells

k-surjective & $fx \xrightarrow{g} fy$ in Y k-cell

$\exists$ $x \xrightarrow{f} y$ k-cell in X & inv. $(k+1)$-cell $fx \cong y$.