# CHANGE OF BASE FOR ENRICHED MODEL CATEGORIES

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Abstract. In this talk we review the change of base theorem for enriched model categories, which states that the change of base of a $\mathcal{V}$-enriched model category along the right adjoint $\mathcal{V} \longrightarrow \mathcal{W}$ of a monoidal Quillen adjunction (whose left adjoint is strong monoidal) is a $\mathcal{W}$-enriched model category. A novelty of our exposition is that we define an enriched category to be a category with extra structure, rather than as an independent structure from which an underlying category is derived. We draw on higher category theory for examples and counterexamples.

In this talk we review the change of base theorem for enriched model categories (see for instance [GM11, Proposition 3.8]):

**Theorem.** *Let $S \dashv T \colon \mathcal{V} \longrightarrow \mathcal{W}$ be a monoidal Quillen adjunction between monoidal model categories. If $(\mathcal{A}, \underline{\mathcal{A}})$ is a model $\mathcal{V}$-category, then $(\mathcal{A}, T_*\underline{\mathcal{A}})$ is a model $\mathcal{W}$-category with the same underlying model category as $(\mathcal{A}, \underline{\mathcal{A}})$.*

An enriched category can be equivalently conceived either as a category with extra structure or as an independent structure from which an underlying category is derived (in [GM11] this distinction is described as being that between thinking of "enriched" as an adjective modifying the noun "category" and thinking of "enriched category" as a noun). While it is the latter point of view that is usually taken in the standard references on enriched category theory (for instance [Kel05]), the former is often more convenient when working with enriched model categories, which are categories equipped with compatible model and enrichment structures [Hov99]. Therefore, for the purposes of this talk, we adopt the following definition of enriched category.

**Definition 1.** Let $(\mathcal{V}, \otimes, I)$ be a monoidal category. A *$\mathcal{V}$-enrichment* $(\underline{\mathcal{A}}, M, j)$ of a category $\mathcal{A}$ consists of the following data:

(i) a functor $\underline{\mathcal{A}}(-,-) \colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{V}$,
(ii) a natural transformation $M \colon \underline{\mathcal{A}}(B, C) \otimes \underline{\mathcal{A}}(A, B) \longrightarrow \underline{\mathcal{A}}(A, C)$, and
(iii) a natural transformation $j \colon I \longrightarrow \underline{\mathcal{A}}(A, A)$,

subject to the usual associativity and left and right unit axioms and to a further *normality* axiom, which states that the composite function

$$\mathcal{A}(A, B) \xrightarrow{\underline{\mathcal{A}}(A,-)} \mathcal{V}(\underline{\mathcal{A}}(A, A), \underline{\mathcal{A}}(A, B)) \xrightarrow{\mathcal{V}(j_A,1)} \mathcal{V}(I, \underline{\mathcal{A}}(A, B)) \tag{1}$$

is a bijection for each pair of objects $A, B \in \mathcal{A}$.

A category $\mathcal{A}$ equipped with a $\mathcal{V}$-enrichment $(\underline{\mathcal{A}}, M, j)$ is called a *$\mathcal{V}$-category*.

Furthermore, we can define a 2-category $\mathcal{V}$-**Cat** of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations, where a $\mathcal{V}$-functor $(F, \underline{F}) \colon (\mathcal{A}, \underline{\mathcal{A}}) \longrightarrow (\mathcal{B}, \underline{\mathcal{B}})$ consists of a functor $F \colon \mathcal{A} \longrightarrow \mathcal{B}$ and a natural transformation $\underline{F} \colon \underline{\mathcal{A}}(A, B) \longrightarrow \underline{\mathcal{B}}(FA, FB)$ subject to the usual composition and unit axioms, and where a $\mathcal{V}$-natural transformation $\theta \colon (F, \underline{F}) \longrightarrow (G, \underline{G})$ is a natural transformation $\theta \colon F \longrightarrow G$ subject to the usual enriched naturality axiom. This 2-category is equivalent to the 2-category of $\mathcal{V}$-categories as it is usually defined (as in [Kel05]; see [Cam18, Corollary 2.14] for a proof of this equivalence). Moreover, there is an evident 2-functor $\mathcal{V}$-**Cat** $\longrightarrow$ **Cat** that sends a $\mathcal{V}$-category $(\mathcal{A}, \underline{\mathcal{A}})$ to its underlying category $\mathcal{A}$.

*Date*: Australian Category Seminar, 14 February 2018.

Recall that a monoidal functor $(T, \varphi, \varphi_0)\colon (\mathcal{V}, \otimes, I) \longrightarrow (\mathcal{W}, \otimes, I)$ between monoidal categories consists of a functor $T\colon \mathcal{V} \longrightarrow \mathcal{W}$, a natural transformation $\varphi\colon TX \otimes TY \longrightarrow T(X \otimes Y)$, and a morphism $\varphi_0\colon I \longrightarrow TI$, subject to associativity and left and right unit axioms. For each monoidal functor $T\colon \mathcal{V} \longrightarrow \mathcal{W}$, there is a "change of base" 2-functor $\mathcal{V}$-**Cat** $\longrightarrow$ $\mathcal{W}$-**Cat** defined in [EK66], which sends each $\mathcal{V}$-category to a $\mathcal{W}$-category. This construction does not preserve underlying categories in general; for the purposes of this talk, we restrict attention to change of base along those monoidal functors that do preserve underlying categories.

**Definition 2.** A monoidal functor $(T, \varphi, \varphi_0)\colon \mathcal{V} \longrightarrow \mathcal{W}$ is said to be *pronormal* if the composite function

$$\mathcal{V}(I, X) \xrightarrow{\;T\;} \mathcal{W}(TI, TX) \xrightarrow{\mathcal{W}(\varphi_0, 1)} \mathcal{W}(I, TX) \tag{2}$$

is a bijection for each object $X \in \mathcal{V}$.

(Note that such monoidal functors are called "normal" in [Kel74], but this conflicts with modern usage.)

**Proposition 3.** *Let $(T, \varphi, \varphi_0)\colon \mathcal{V} \longrightarrow \mathcal{W}$ be a pronormal monoidal functor. For each $\mathcal{V}$-enrichment $\underline{\mathcal{A}} = (\underline{\mathcal{A}}, M, j)$ of a category $\mathcal{A}$, the following data define a $\mathcal{W}$-enrichment $T_*\underline{\mathcal{A}}$ of $\mathcal{A}$, called the* change of base of $\underline{\mathcal{A}}$ along $T$*:*

(i) *hom-functor*

$$\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \xrightarrow{\underline{\mathcal{A}}(-,-)} \mathcal{V} \xrightarrow{\;T\;} \mathcal{W},$$

(ii) *composition natural transformation*

$$T\underline{\mathcal{A}}(B, C) \otimes T\underline{\mathcal{A}}(A, B) \xrightarrow{\;\varphi\;} T(\underline{\mathcal{A}}(B, C) \otimes \underline{\mathcal{A}}(A, B)) \xrightarrow{TM} T\underline{\mathcal{A}}(A, C),$$

(iii) *unit natural transformation*

$$I \xrightarrow{\varphi_0} TI \xrightarrow{Tj} T\underline{\mathcal{A}}(A, A).$$

*Moreover, this defines the action on objects of a 2-functor $T_*\colon \mathcal{V}$-**Cat** $\longrightarrow$ $\mathcal{W}$-**Cat** that commutes with the underlying 2-functors to* **Cat**.

*Proof.* The proof follows that of [EK66, Proposition II.6.3]. It remains only to verify the normality axiom (1) for $T_*\underline{\mathcal{A}}$. This is proved by the following commutative diagram,

$$\begin{array}{ccccc}
\mathcal{A}(A,B) & \xrightarrow{\underline{\mathcal{A}}(A,-)} & \mathcal{V}(\underline{\mathcal{A}}(A,A), \underline{\mathcal{A}}(A,B)) & \xrightarrow{\;T\;} & \mathcal{W}(T\underline{\mathcal{A}}(A,A), T\underline{\mathcal{A}}(A,B)) \\
& \searrow^{\cong} & \downarrow{\scriptstyle \mathcal{V}(j,1)} & & \downarrow{\scriptstyle \mathcal{W}(Tj,1)} \\
& & \mathcal{V}(I, \underline{\mathcal{A}}(A,B)) & \xrightarrow{\;T\;} & \mathcal{W}(TI, T\underline{\mathcal{A}}(A,B)) \\
& & & \searrow^{\cong} & \downarrow{\scriptstyle \mathcal{W}(\varphi_0,1)} \\
& & & & \mathcal{W}(I, T\underline{\mathcal{A}}(A,B))
\end{array}$$

whose rectangular region commutes by functoriality of $T$, and whose two diagonal composites are isomorphisms by the assumptions of normality of $\underline{\mathcal{A}}$ and pronormality of $T$. $\square$

We will see in Theorem 8 that change of base is best behaved along the right adjoint of a monoidal adjunction between biclosed monoidal categories. A *monoidal adjunction* is defined to be an adjunction in the 2-category of monoidal categories, monoidal functors, and monoidal natural transformations [EK66]. Note that the left adjoint of a monoidal adjunction is necessarily strong monoidal, meaning that its monoidal constraints $\varphi$ and $\varphi_0$ are invertible; this is an instance of doctrinal adjunction [Kel74]. In particular, we have the following result [Kel74, Proposition 2.1].

**Proposition 4.** *The right adjoint of a monoidal adjunction is pronormal.*

*Proof.* Let $S \dashv T\colon \mathcal{V} \longrightarrow \mathcal{W}$ be a monoidal adjunction with unit $\eta\colon 1_{\mathcal{W}} \longrightarrow TS$. Consider the following commutative diagram,

$$\begin{array}{ccccc} \mathcal{V}(SI,X) & \xrightarrow{T} & \mathcal{W}(TSI,TX) & \xrightarrow{\mathcal{W}(\eta_I,1)} & \mathcal{W}(I,TX) \\ {\scriptstyle \mathcal{V}(\varphi_0,1)}\downarrow & & \downarrow{\scriptstyle \mathcal{W}(T\varphi_0,1)} & & \| \\ \mathcal{V}(I,X) & \xrightarrow[T]{} & \mathcal{W}(TI,TX) & \xrightarrow[\mathcal{W}(\varphi_0,1)]{} & \mathcal{W}(I,TX) \end{array}$$

whose left-hand region commutes by functoriality of $T$ and whose right-hand region commutes since $\eta\colon 1_{\mathcal{W}} \longrightarrow TS$ is a monoidal natural transformation. By definition, $T$ is pronormal if the bottom composite is a bijection; this follows because the top composite is a bijection by adjointness and the left-hand vertical arrow is a bijection since $S$ is strong monoidal. □

Hence we can define change of base along the right adjoint of a monoidal adjunction as in Proposition 3.

A monoidal category is said to be *biclosed* if its tensor product functor has a right adjoint in each variable.

$$\mathcal{V}(Y,\langle X,Z\rangle) \cong \mathcal{V}(X\otimes Y,Z) \cong \mathcal{V}(X,[Y,Z]) \tag{3}$$

In particular, for a $\mathcal{V}$-category $(\mathcal{A},\underline{\mathcal{A}})$, the composition morphisms

$$\underline{\mathcal{A}}(B,C)\otimes\underline{\mathcal{A}}(A,B) \xrightarrow{M} \underline{\mathcal{A}}(A,C)$$

correspond under the bijections (3) to the morphisms which we denote as follows.

$$\underline{\mathcal{A}}(B,C) \xrightarrow{\underline{\mathcal{A}}(A,-)} [\underline{\mathcal{A}}(A,B),\underline{\mathcal{A}}(A,C)] \qquad \underline{\mathcal{A}}(A,B) \xrightarrow{\underline{\mathcal{A}}(-,C)} \langle\underline{\mathcal{A}}(B,C),\underline{\mathcal{A}}(A,C)\rangle$$

**Definition 5.** Let $\mathcal{V}$ be a biclosed monoidal category. Recall (for instance from [Kel69]) that a $\mathcal{V}$-category $\underline{\mathcal{A}}$ is *tensored* if for each pair of objects $X\in V$ and $A\in\mathcal{A}$, there exists an object $X*A\in\mathcal{A}$ and a morphism $d\colon X\longrightarrow\underline{\mathcal{A}}(A,X*A)$ in $\mathcal{V}$ such that the composite morphism

$$\underline{\mathcal{A}}(X*A,B) \xrightarrow{\underline{\mathcal{A}}(A,-)} [\underline{\mathcal{A}}(A,X*A),\underline{\mathcal{A}}(A,B)] \xrightarrow{[d,1]} [X,\underline{\mathcal{A}}(A,B)] \tag{4}$$

is an isomorphism in $\mathcal{V}$ for every object $B\in\mathcal{A}$.

A $\mathcal{V}$-category $(\mathcal{A},\underline{\mathcal{A}})$ is *cotensored* if its opposite $\mathcal{V}^{\mathrm{rev}}$-category $(\mathcal{A}^{\mathrm{op}},\underline{\mathcal{A}}^{\mathrm{op}})$ is tensored, i.e. if for each pair of objects $X\in\mathcal{V}$ and $B\in\mathcal{A}$, there exists an object $X\pitchfork B\in\mathcal{A}$ and a morphism $c\colon X\longrightarrow\underline{\mathcal{A}}(X\pitchfork B,B)$ such that the composite morphism

$$\underline{\mathcal{A}}(A,X\pitchfork B) \xrightarrow{\underline{\mathcal{A}}(-,B)} \langle\underline{\mathcal{A}}(X\pitchfork B,B),\underline{\mathcal{A}}(A,B)\rangle \xrightarrow{\langle c,1\rangle} \langle X,\underline{\mathcal{A}}(A,B)\rangle$$

is an isomorphism in $\mathcal{W}$ for every object $A\in\mathcal{A}$.

The following well known theorem goes back in part to [Kel69, §5]. First we need a lemma. [Recall that the 2-category of closed monoidal categories is equivalent to the 2-category of monoidal closed categories.]

**Lemma 6.** *Let $S\dashv T\colon\mathcal{V}\longrightarrow\mathcal{W}$ be a monoidal adjunction between closed monoidal categories. For each pair of objects $X\in V$ and $Y\in W$, the composite morphism*

$$T[SY,X] \xrightarrow{\psi} [TSY,TX] \xrightarrow{[\eta,1]} [Y,TX] \tag{5}$$

*is an isomorphism in $\mathcal{W}$.*

*Proof.* We show that the composite

$$[Y,TX] \xrightarrow{\eta} TS[Y,TX] \xrightarrow{T\psi} T[SY,STX] \xrightarrow{T[1,\varepsilon]} T[SY,X]$$

is inverse to (5). This is shown by the commutativity of the following diagrams, where we use that $\eta\colon 1\longrightarrow TS$ and $\varepsilon\colon ST\longrightarrow 1$ are closed natural transformations. Observe that the triangle identities of the adjunction $S \dashv T$ imply that certain sides of the following diagrams are identities.

$$\begin{array}{ccccc}
T[SY,X] & \xrightarrow{\psi} & [TSY,TX] & \xrightarrow{[\eta,1]} & [Y,TX] \\
{\scriptstyle\eta}\downarrow & & {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\
TST[SY,X] & \xrightarrow{TS\psi} & TS[TSY,TX] & \xrightarrow{TS[\eta,1]} & TS[Y,TX] \\
\Big\downarrow & & {\scriptstyle T\psi}\downarrow & & \downarrow{\scriptstyle T\psi} \\
{\scriptstyle T\varepsilon} & & T[STSY,STX] & \xrightarrow{T[S\eta,1]} & T[SY,STX] \\
\Big\downarrow & & {\scriptstyle T[1,\varepsilon]}\downarrow & & \downarrow{\scriptstyle T[1,\varepsilon]} \\
T[SY,X] & \xrightarrow[T[\varepsilon,1]]{} & T[STSY,X] & \xrightarrow[T[S\eta,1]]{} & T[SY,X]
\end{array}$$

$$\begin{array}{ccccc}
TS[Y,TX] & \xrightarrow{T\psi} & T[SY,STX] & \xrightarrow{T[1,\varepsilon]} & T[SY,X] \\
\Big\uparrow & & {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\
{\scriptstyle\eta} & & [TSY,TSTX] & \xrightarrow{[1,T\varepsilon]} & [TSY,TX] \\
\Big| & & {\scriptstyle[\eta,1]}\downarrow & & \downarrow{\scriptstyle[\eta,1]} \\
[Y,TX] & \xrightarrow[{[1,\eta]}]{} & [Y,TSTX] & \xrightarrow[{[1,T\varepsilon]}]{} & [Y,TX]
\end{array}$$

$\square$

**Remark 7.** This lemma proves moreover that the monoidal adjunction $S \dashv T \colon \mathcal{V} \longrightarrow \mathcal{W}$ between closed monoidal categories induces a $\mathcal{W}$-enriched adjunction $\check{S} \dashv \widehat{T} \colon (\mathcal{V}, T_*\underline{\mathcal{V}}) \longrightarrow (\mathcal{W}, \underline{\mathcal{W}})$, where $(\mathcal{U}, \underline{\mathcal{U}})$ denotes the self-enrichment of a closed monoidal category $\mathcal{U}$.

**Theorem 8.** *Let $S \dashv T \colon \mathcal{V} \longrightarrow \mathcal{W}$ be a monoidal adjunction between biclosed monoidal categories. If $(\mathcal{A}, \underline{\mathcal{A}})$ is a tensored and cotensored $\mathcal{V}$-category, then $(\mathcal{A}, T_*\underline{\mathcal{A}})$ is a tensored and cotensored $\mathcal{W}$-category, with tensors $Y * A = SY * A$ and cotensors $Y \pitchfork B = SY \pitchfork B$.*

*Proof.* We first prove that $(\mathcal{A}, T_*\underline{\mathcal{A}})$ is a tensored $\mathcal{W}$-category. The proof that it is cotensored then follows by duality, since change of base commutes with taking opposite categories (note that the monoidal adjunction $S \dashv T \colon \mathcal{V} \longrightarrow \mathcal{W}$ induces a monoidal adjunction $S^{\mathrm{rev}} \dashv T^{\mathrm{rev}} \colon \mathcal{V}^{\mathrm{rev}} \longrightarrow \mathcal{W}^{\mathrm{rev}}$ between their reverse duals.)

Let $Y \in \mathcal{W}$ and $A \in \mathcal{A}$. We show that $SY * A$ defines a $\mathcal{W}$-enriched tensor product in $(\mathcal{A}, T_*\underline{\mathcal{A}})$ with unit given by the following composite morphism in $\mathcal{W}$.

$$Y \xrightarrow{\eta} TSY \xrightarrow{Td} T\underline{\mathcal{A}}(A, SY * A)$$

This is proved by the following commutative diagram,

$$\begin{array}{ccccc}
T\underline{\mathcal{A}}(SY * A, B) & \xrightarrow{T\underline{\mathcal{A}}(A,-)} & T[\underline{\mathcal{A}}(A, SY * A), \underline{\mathcal{A}}(A, B)] & \xrightarrow{\psi} & [T\underline{\mathcal{A}}(A, SY * A), T\underline{\mathcal{A}}(A, B)] \\
& \searrow^{\cong} & \downarrow{\scriptstyle T[d,1]} & & \downarrow{\scriptstyle [Td,1]} \\
& & T[SY, \underline{\mathcal{A}}(A, B)] & \xrightarrow{\psi} & [TSY, T\underline{\mathcal{A}}(A, B)] \\
& & & \searrow^{\cong} & \downarrow{\scriptstyle [\eta,1]} \\
& & & & [Y, T\underline{\mathcal{A}}(A, B)]
\end{array}$$

whose rectangular region commutes by naturality of the closed functor constraint $\psi$, and whose diagonal composites are isomorphisms by the universal property (4) of the tensor product $SY * A$ in the $\mathcal{V}$-category $(\mathcal{A}, \underline{\mathcal{A}})$ and by Lemma 6. $\square$

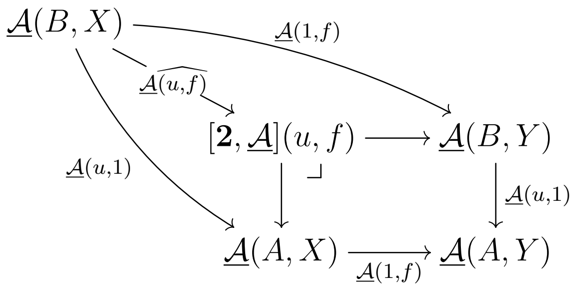
**Remark 9.** One can give an alternative proof of Theorem 8 using Remark 7 and the fact that a $\mathcal{V}$-category $(\mathcal{A}, \underline{\mathcal{A}})$ is tensored if and only if every representable $\mathcal{V}$-functor $\underline{\mathcal{A}}(A, -) \colon (\mathcal{A}, \underline{\mathcal{A}}) \longrightarrow (\mathcal{V}, \underline{\mathcal{V}})$ has a $\mathcal{V}$-enriched left adjoint, or by using the enriched Yoneda lemma and the fact that the left adjoint of a monoidal adjunction is strong monoidal.

We refer to [Hov99] for the notions of monoidal model category and monoidal Quillen adjunction, and for the notion of enriched model category, which we now recall.

**Definition 10.** Let $\mathcal{V}$ be a monoidal model category and let $\mathcal{A}$ be a complete and cocomplete category. A *$\mathcal{V}$-enriched model structure* on $\mathcal{A}$ consists of:

(i) a model structure on $\mathcal{A}$,
(ii) a tensored and cotensored $\mathcal{V}$-enrichment $\underline{\mathcal{A}}$ of $\mathcal{A}$,

such that for each cofibration $u \colon A \longrightarrow B$ and fibration $f \colon X \longrightarrow Y$ in $\mathcal{A}$, the Leibniz hom $\widehat{\underline{\mathcal{A}}(u,f)}$ is a fibration in $\mathcal{V}$, and is moreover a trivial fibration if either $u$ or $f$ is a weak equivalence.




A complete and cocomplete category equipped with a $\mathcal{V}$-enriched model structure is called a *$\mathcal{V}$-enriched model category* or simply a *model $\mathcal{V}$-category.*

We now prove the change of base theorem for enriched model categories, with which we opened this talk.

**Theorem 11.** *Let $S \dashv T \colon \mathcal{V} \longrightarrow \mathcal{W}$ be a monoidal Quillen adjunction between monoidal model categories. If $(\mathcal{A}, \underline{\mathcal{A}})$ is a model $\mathcal{V}$-category, then $(\mathcal{A}, T_*\underline{\mathcal{A}})$ is a model $\mathcal{W}$-category with the same underlying model category as $(\mathcal{A}, \underline{\mathcal{A}})$.*

*Proof.* By Theorem 8, $(\mathcal{A}, T_*\underline{\mathcal{A}})$ is a tensored and cotensored $\mathcal{W}$-category with the same underlying category $\mathcal{A}$ as $(\mathcal{A}, \underline{\mathcal{A}})$, which by assumption is complete and cocomplete and equipped with a model structure. Hence it remains to show that this model structure is compatible with the $\mathcal{W}$-enrichment $T_*\underline{\mathcal{A}}$ defined by change of base along $T$.

Let $u \colon A \longrightarrow B$ be a cofibration and $f \colon X \longrightarrow Y$ a fibration in $\mathcal{A}$. By the definition of the $\mathcal{W}$-enrichment $T_*\underline{\mathcal{A}}$, and since $T$ preserves pullbacks, we have that the Leibniz hom $\widehat{(T_*\underline{\mathcal{A}})(u,f)}$ is isomorphic in the arrow category $\mathcal{W}^{\mathbf{2}}$ to $T(\widehat{\underline{\mathcal{A}}(u,f)})$. But the Leibniz hom $\widehat{\underline{\mathcal{A}}(u,f)}$ is a fibration in $\mathcal{V}$, and is a trivial fibration if either $u$ or $f$ is a weak equivalence, since $(\mathcal{A}, \underline{\mathcal{A}})$ is a model $\mathcal{V}$-category. Hence $\widehat{(T_*\underline{\mathcal{A}})(u,f)}$ is a fibration in $\mathcal{W}$, and is a trivial fibration if either $u$ or $f$ is a weak equivalence in $\mathcal{A}$, since $T$ preserves fibrations and trivial fibrations. Therefore $(\mathcal{A}, T_*\underline{\mathcal{A}})$ is a model $\mathcal{W}$-category. □

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