# The model category of algebraically cofibrant 2-categories

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# Two-dimensional category theory

**Coherence for bicategories**

Every bicategory is biequivalent to a 2-category.

Moreover, one can model the **category theory** of bicategories by "2-category theory":

**Lack, *A 2-categories companion*:**

> *2-category theory is a "middle way" between* **Cat***-category theory and bicategory theory. It uses enriched category theory, but not in the simple minded way of* **Cat***-category theory; and it cuts through some of the technical nightmares of bicategories.*

This could also be described as "homotopy coherent" **Cat**-category theory; we enrich over **Cat** not merely as a monoidal category, but as a monoidal category with inherent higher structure: **Cat** as a **monoidal model category**.

One dimension higher:

**Theorem (Gordon–Power–Street)**

*Every tricategory is triequivalent to a* **Gray***-category.*

**Gray** denotes the category **2-Cat** equipped with Gray's symmetric monoidal closed structure.

**2-Cat** is a monoidal model category with respect to this monoidal structure and Lack's model structure.

To a large extent, one can model the category theory of tricategories by "homotopy coherent" **Gray**-category theory.

# A fundamental obstruction

However, there is a fundamental obstruction to the development of a *purely* **Gray**-enriched model for three-dimensional category theory:

Not every 2-category is **cofibrant** in Lack's model structure.

In practice, the result is that certain basic constructions fail to define **Gray**-functors; they are at best "locally weak **Gray**-functors".

This obstruction can be overcome by the introduction of a new base for enrichment: the monoidal model category $\mathbf{2\text{-}Cat}_Q$ of **algebraically cofibrant** 2-**categories**, which is the subject of this talk.

We will see that:

- Every object of $\mathbf{2\text{-}Cat}_Q$ is cofibrant.
- $\mathbf{2\text{-}Cat}_Q$ is monoidally Quillen equivalent to $\mathbf{2\text{-}Cat}$.

# Plan

1. The category of algebraically cofibrant 2-categories
2. The left-induced model structure
3. Bicategories as fibrant objects
4. Monoidal structures
5. A counterexample

# Plan

# The category of algebraically cofibrant 2-categories

Let $Q$ denote the **normal pseudofunctor classifier** comonad on **2-Cat**.

$$\begin{tikzcd} \mathbf{2\text{-}Cat} \arrow[r, phantom, "\perp"] & \mathbf{2\text{-}Cat}_{\mathbf{nps}} \arrow[l, bend right, "Q"'] \arrow[l, leftarrow, bend left] \end{tikzcd}$$

$$\frac{QA \longrightarrow B \qquad \text{2-functors}}{A \rightsquigarrow B \qquad \text{normal pseudofunctors}}$$

The 2-category $QA$ can be constructed by taking the (boba, loc ff) factorisation of the "composition" 2-functor $PUA \longrightarrow A$. ($PUA =$ the free category on the underlying reflexive graph of $A$)

$$PUA \xrightarrow{\text{boba}} QA \xrightarrow{\text{loc ff}} A$$

### The coalgebraic definition of $\mathbf{2\text{-}Cat}_Q$

Define $\mathbf{2\text{-}Cat}_Q$ to be the category of coalgebras for the normal pseudofunctor classifier comonad $Q$ on **2-Cat**.

A 2-category admits **at most one** $Q$-coalgebra structure, and does so if and only if it is **cofibrant**, i.e. its underlying category is **free**.

# The category of free categories I

## Definition (atomic morphism)

A morphism $f$ in a category is **atomic** if:

(i) $f$ is not an identity, and

(ii) if $f = hg$, then $g$ is an identity or $h$ is an identity.

## Definition (free category)

A category $C$ is **free** if every morphism $f$ in $C$ can be uniquely expressed as a composite of atomic morphisms $(n \geq 0,\ f = f_n \circ \cdots f_1)$.

## Definition (morphism of free categories)

A functor $C \longrightarrow D$ between free categories is a **morphism of free categories** if it sends each atomic morphism in $C$ to an atomic morphism or an identity morphism in $D$.

These objects and morphisms form the **category of free categories**.

# The category of free categories II

Let **Gph** denote the category of **reflexive graphs**. Recall the free-forgetful adjunction:

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{Gph} \arrow[l, bend right, "F"'] \end{tikzcd}$$

Write $P = FU$ for the induced comonad on **Cat**. A category admits at most one $P$-coalgebra structure, and does so if and only if it is free.

**Proposition**

*The following three categories are isomorphic.*

1. *The category of free categories and their morphisms.*
2. *The replete image of the (pseudomonic) functor $F\colon \mathbf{Gph} \longrightarrow \mathbf{Cat}$.*
3. *The category $\mathbf{Cat}_P$ of coalgebras for the comonad $P$ on $\mathbf{Cat}$.*

Furthermore, each of these categories is equivalent to the category **Gph** of reflexive graphs.

# The category of cofibrant 2-categories

**Definition (cofibrant 2-category)**

A 2-category is **cofibrant** if its underlying category is free.

**Definition (morphism of cofibrant 2-categories)**

A 2-functor between cofibrant 2-categories is a **morphism of cofibrant 2-categories** if its underlying functor is a morphism of free categories.

**Proposition (the elementary definition of $\mathbf{2\text{-}Cat}_Q$)**

*The category $\mathbf{2\text{-}Cat}_Q$ is isomorphic to the (replete, non-full) subcategory of $\mathbf{2\text{-}Cat}$ consisting of the cofibrant 2-categories and their morphisms.*

The comonadic functor $V\colon \mathbf{2\text{-}Cat}_Q \longrightarrow \mathbf{2\text{-}Cat}$ is the replete subcategory inclusion.

# $\mathbf{2\text{-}Cat}_Q$ as an iso-comma category

Thus the category $\mathbf{2\text{-}Cat}_Q$ is the pullback:

$$\begin{tikzcd} \mathbf{2\text{-}Cat}_Q \arrow[r, "V"] \arrow[d, "U"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{2\text{-}Cat} \arrow[d, "U"] \\ \mathbf{Cat}_P \arrow[r, "V"'] & \mathbf{Cat} \end{tikzcd}$$

Furthermore, $\mathbf{2\text{-}Cat}_Q$ is equivalent to the iso-comma category

$$\begin{tikzcd} \mathbf{Gph} \downarrow_{\cong} \mathbf{2\text{-}Cat} \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathbf{2\text{-}Cat} \arrow[d, "U"] \\ \mathbf{Gph} \arrow[r, "F"'] & \mathbf{Cat} \end{tikzcd}$$

in which an object $(X, A, \varphi)$ consists of a reflexive graph $X$, a 2-category $A$, and a boba 2-functor $\varphi\colon FX \longrightarrow A$.

# Categorical properties of $\mathbf{2\text{-}Cat}_Q$

It is immediate from either definition that:

**Observation**

The inclusion functor $V\colon \mathbf{2\text{-}Cat}_Q \longrightarrow \mathbf{2\text{-}Cat}$

- is pseudomonic (i.e. faithful, and full on isomorphisms),
- creates colimits,
- has a right adjoint.

$$\begin{tikzcd} \mathbf{2\text{-}Cat} \arrow[r, bend right, "Q"'] \arrow[r, phantom, "\perp"] & \mathbf{2\text{-}Cat}_Q \arrow[l, bend right, "V"'] \end{tikzcd}$$

Moreover, it is not difficult to prove that:

**Proposition**

*The category $\mathbf{2\text{-}Cat}_Q$ is locally finitely presentable.*

# Plan

# Lack's model structure for 2-categories

The goal of this section is to prove that $\mathbf{2\text{-}Cat}_Q$ admits a model structure **left-induced** from **Lack's model structure for** 2-**categories** along the inclusion $V\colon \mathbf{2\text{-}Cat}_Q \longrightarrow \mathbf{2\text{-}Cat}$.

## Lack's model structure on **2-Cat**

Lack constructed a model structure on **2-Cat** in which a 2-functor $F\colon A \longrightarrow B$ is:

- a **weak equivalence** iff it is a **biequivalence**, i.e. is surjective on objects up to equivalence, and is an equivalence on hom-categories;
- a **fibration** iff it is an **equifibration**, i.e. has the equivalence lifting property, and is an isofibration on hom-categories;
- a **trivial fibration** iff it is surjective on objects, and is a surjective equivalence on hom-categories.

Every 2-category is fibrant in this model structure.
A 2-category is cofibrant in this model structure if and only if it is a cofibrant 2-category.

# The left-induced model structure

The goal of this section is to prove that $\mathbf{2\text{-}Cat}_Q$ admits a model structure in which a morphism of cofibrant 2-categories is:

- a **cofibration** iff it is a cofibration in Lack's model structure on $\mathbf{2\text{-}Cat}$,
- a **weak equivalence** iff it is a weak equivalence in Lack's model structure on $\mathbf{2\text{-}Cat}$ (i.e. a biequivalence).

**Nec. & suff. conditions for existence of the left-induced model structure**

The left-induced model structure on $\mathbf{2\text{-}Cat}_Q$ exists if and only if

1. the cofibrations in $\mathbf{2\text{-}Cat}_Q$ form the left class of a wfs on $\mathbf{2\text{-}Cat}_Q$,
2. the trivial cofibrations in $\mathbf{2\text{-}Cat}_Q$ form the left class of a wfs on $\mathbf{2\text{-}Cat}_Q$, and
3. the **acyclicity condition** holds: in $\mathbf{2\text{-}Cat}_Q$, any morphism with the RLP wrt all cofibrations is a biequivalence.

# Cofibrations

In general, the cofibrations in Lack's model structure on $\mathbf{2\text{-}Cat}_Q$ are difficult to describe explicitly. However:

**Proposition**

*Let $F \colon A \longrightarrow B$ be a 2-functor between cofibrant 2-categories. Then the following are equivalent.*

(i) *$F$ is a cofibration in Lack's model structure on* **2-Cat**.

(ii) *The underlying functor of $F$ is free on a monomorphism of reflexive graphs.*

Hence every cofibration in **2-Cat** between cofibrant 2-categories is a morphism of cofibrant 2-categories.

# Trivial fibrations

## The (monomorphism, trivial fibration) wfs on **Gph**

A morphism of reflexive graphs is said to be a **trivial fibration** if it is surjective on objects and full.
The classes (monomorphism, trivial fibration) form a wfs on the category **Gph** of reflexive graphs.

## Definition (trivial fibration in $\mathbf{2\text{-}Cat}_Q$)

A morphism of cofibrant 2-categories is a **trivial fibration** (as a morphism in $\mathbf{2\text{-}Cat}_Q$) if

1. its underlying functor is free on a trivial fibration of reflexive graphs,
2. it is locally fully faithful.

## Proposition

*If a morphism of cofibrant 2-quasi-categories is a trivial fibration (as a morphism in* $\mathbf{2\text{-}Cat}_Q$*), then it is a biequivalence.*

## Proposition

*The classes (cofibration, trivial fibration) form a (cofibrantly generated) weak factorisation system on* $\mathbf{2\text{-}Cat}_Q$.

## Proof.

Construct factorisations and diagonal fillers using:

- the equivalence of categories $\mathbf{2\text{-}Cat}_Q \simeq \mathbf{Gph} \downarrow_{\cong} \mathbf{2\text{-}Cat}$,
- the (monomonorphism, trivial fibration) wfs on **Gph**, and
- the (boba, loc ff) factorisation system on **2-Cat**. □

This is condition (1) for the existence of the left-induced model structure. We can also deduce condition (3).

## Corollary (acyclicity condition)

*In* $\mathbf{2\text{-}Cat}_Q$*, any morphism with the RLP wrt all cofibrations is a trivial fibration (in* $\mathbf{2\text{-}Cat}_Q$*), and hence a biequivalence.*

# The (trivial cofibration, fibration) wfs

**Proposition**

*The trivial cofibrations in $\mathbf{2\text{-}Cat}_Q$ form the left class of a (cofibrantly generated) wfs on $\mathbf{2\text{-}Cat}_Q$.*

**Proof.**

In $\mathbf{2\text{-}Cat}$, the trivial cofibrations and fibrations for Lack's model structure on $\mathbf{2\text{-}Cat}$ form a cofibrantly generated wfs.
The inclusion $\mathbf{2\text{-}Cat}_Q \longrightarrow \mathbf{2\text{-}Cat}$ is a left adjoint functor between locally (finitely) presentable categories.
A theorem of Makkai–Rosický then implies that the trivial cofibrations in $\mathbf{2\text{-}Cat}_Q$ form the left class of a (cofibrantly generated) wfs on $\mathbf{2\text{-}Cat}_Q$. $\square$

**Theorem (existence of the left-induced model structure)**

*There exists a (combinatorial) model structure on $\mathbf{2\text{-}Cat}_Q$ whose cofibrations and weak equivalences are created by the inclusion functor $\mathbf{2\text{-}Cat}_Q \longrightarrow \mathbf{2\text{-}Cat}$ from Lack's model structure for 2-categories.*

# A Quillen equivalence

**Theorem**

*The adjunction*

$$\begin{tikzcd} \mathbf{2\text{-}Cat} \arrow[r, bend right, "Q"'] \arrow[r, phantom, "\perp"] & \mathbf{2\text{-}Cat}_Q \arrow[l, bend right, "V"'] \end{tikzcd}$$

*is a Quillen equivalence between Lack's model structure on* **2-Cat** *and the left-induced model structure on* $\mathbf{2\text{-}Cat}_Q$.

**Proof.**

By definition of the model structure on $\mathbf{2\text{-}Cat}_Q$, the left adjoint preserves cofibrations, and preserves and reflects weak equivalences.

For each 2-category $A$, the counit morphism $QA \longrightarrow A$ is a weak equivalence in **2-Cat**. $\square$

# Plan

# Fibrant objects

The functor $Q\colon \mathbf{2\text{-}Cat} \longrightarrow \mathbf{2\text{-}Cat}_Q$ is a right Quillen functor.
Hence, for every 2-category $A$, $QA$ is a fibrant object in $\mathbf{2\text{-}Cat}_Q$.

## Proposition

*A cofibrant 2-category is a fibrant object in the left-induced model structure on $\mathbf{2\text{-}Cat}_Q$ if and only if it is a retract in $\mathbf{2\text{-}Cat}_Q$ of the normal pseudofunctor classifier $QA$ of some 2-category $A$.*

## Proof.

Sufficiency: A retract of a fibrant object is fibrant.
Necessity: For every cofibrant 2-category $A$, the $Q$-coalgebra structure map $\alpha\colon A \longrightarrow QA$ is a trivial cofibration in $\mathbf{2\text{-}Cat}_Q$.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\alpha"'] & A \\ QA \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

$\square$

# The full subcategory of fibrant objects

The full image of the functor $Q\colon \mathbf{2\text{-}Cat} \longrightarrow \mathbf{2\text{-}Cat}_Q$ is the category $\mathbf{2\text{-}Cat_{nps}}$ of 2-categories and normal pseudofunctors ($=$ the Kleisli category for the comonad $Q$ on $\mathbf{2\text{-}Cat}$).

$$\mathbf{2\text{-}Cat}_Q(QA, QB) \cong \mathbf{2\text{-}Cat}(QA, B) \cong \mathbf{2\text{-}Cat_{nps}}(A, B)$$

So we have a functor $Q\colon \mathbf{2\text{-}Cat_{nps}} \longrightarrow (\mathbf{2\text{-}Cat}_Q)_{\mathrm{fib}}$ which is

- fully faithful, and
- surjective on objects up to retracts.

Hence this functor witnesses $(\mathbf{2\text{-}Cat}_Q)_{\mathrm{fib}}$ as the **Cauchy completion** of $\mathbf{2\text{-}Cat_{nps}}$.
But the Cauchy completion of $\mathbf{2\text{-}Cat_{nps}}$ is none other than $\mathbf{Bicat_{nps}}$.

**Theorem**

*The* **normal strictification** *functor* $Q\colon \mathbf{Bicat_{nps}} \longrightarrow \mathbf{2\text{-}Cat}_Q$ *is fully faithful, and its essential image consists of the fibrant objects for the left-induced model structure.*

# Intrinsic characterisation of fibrant objects

**Theorem**

*Let $A$ be a cofibrant 2-category. Then the following are equivalent.*

(i) *$A$ is a fibrant object in the left-induced model structure on $\mathbf{2\text{-}Cat}_Q$.*

(ii) *$A \cong QB$ for some bicategory $B$.*

(iii) *Every non-identity morphism in $A$ is isomorphic (via an invertible 2-cell) to an atomic morphism in $A$.*

(iv) *$A$ has the RLP in $\mathbf{2\text{-}Cat}_Q$ wrt $\mathbf{3} \longrightarrow Q\mathbf{3}$.*

**Proof.**

The step (iii) $\Rightarrow$ (ii) uses two-dimensional monad theory. □

# Fibrations between fibrant objects

**Theorem**

*Let $F\colon A \longrightarrow B$ be a normal pseudofunctor between bicategories. Then the following are equivalent.*

(i) *$QF\colon QA \longrightarrow QB$ is a fibration in the left-induced model structure on $\mathbf{2\text{-}Cat}_Q$.*

(ii) *$F\colon A \longrightarrow B$ is an equifibration, i.e. has the equivalence lifting property and is an isofibration on hom-categories.*

This theorem characterises the fibrations with fibrant codomain in $\mathbf{2\text{-}Cat}_Q$.

I do not have an explicit description of the fibrations in $\mathbf{2\text{-}Cat}_Q$ with arbitrary codomain.

**Remark**

The left-induced model structure on $\mathbf{2\text{-}Cat}_Q$ is not right proper.

# Plan

# The Gray monoidal structure

The (symmetric) Gray tensor product of two cofibrant 2-categories is cofibrant. Also, $\mathbf{1}$ is cofibrant.
Since the inclusion $\mathbf{2\text{-}Cat}_Q \longrightarrow \mathbf{2\text{-}Cat}$ is full on isomorphisms, Gray's symmetric monoidal structure on $\mathbf{2\text{-}Cat}$ restricts to one on $\mathbf{2\text{-}Cat}_Q$.
By the adjoint functor theorem (or by direct construction), this symmetric monoidal structure on $\mathbf{2\text{-}Cat}_Q$ is **closed**.

**Theorem**

$\mathbf{2\text{-}Cat}_Q$ *is a* **monoidal model category** *wrt the Gray monoidal structure and the left-induced model structure. The adjunction* $V \dashv Q \colon \mathbf{2\text{-}Cat} \longrightarrow \mathbf{2\text{-}Cat}_Q$ *is a monoidal Quillen equivalence.*

If $A$ and $B$ are bicategories, then $[QA, QB] = Q\mathbf{Hom}(A, B)$.

A category enriched over $\mathbf{2\text{-}Cat}_Q$ (with the Gray monoidal structure) is a "locally cofibrant **Gray**-category". E.g. $\mathbf{Bicat}_{\mathbf{nps}}$ underlies a locally cofibrant **Gray**-category with homs as above.

Unlike Lack's model structure on **2-Cat**, the left-induced model structure on $\mathbf{2\text{-}Cat}_Q$ is also cartesian.

**Theorem**

*The category* $\mathbf{2\text{-}Cat}_Q$ *is cartesian closed, and is a* **cartesian model category** *wrt the left-induced model structure.*

The full embedding $Q\colon \mathbf{Bicat_{nps}} \longrightarrow \mathbf{2\text{-}Cat}_Q$ is a cartesian closed functor.

Let $A$ and $B$ be cofibrant 2-categories, and let $FX \longrightarrow A$ and $FY \longrightarrow B$ be boba 2-functors. The cartesian product $A \boxtimes B$ in $\mathbf{2\text{-}Cat}_Q$ can be constructed via the (boba, loc ff) factorisation of $F(X \times Y) \longrightarrow A \times B$

$$F(X \times Y) \xrightarrow{\text{boba}} A \boxtimes B \xrightarrow{\text{loc ff}} A \times B$$

$$\mathbf{2} \otimes \mathbf{2} \quad = \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \qquad ; \qquad \mathbf{2} \boxtimes \mathbf{2} \quad = \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

# Plan

# Model categories of algebraically cofibrant objects?

Nikolaus and Bourke have shown that, if $\mathcal{M}$ is a "nice" (e.g. combinatorial) model category, then for any **fibrant replacement monad** $T$ on $\mathcal{M}$, the category $\mathcal{M}^T$ of $T$-algebras ("algebraically fibrant objects") admits a model structure, right-induced from $\mathcal{M}$ along the forgetful functor $U\colon \mathcal{M}^T \longrightarrow \mathcal{M}$.

It has been asked (Ching–Riehl, Bourke): for any combinatorial model category $\mathcal{M}$ and any **cofibrant replacement comonad** $G$ on $\mathcal{M}$, does the category of $G$-coalgebras ("algebraically cofibrant objects") admit a model structure left-induced from $\mathcal{M}$ along the forgetful functor $\mathcal{M}_G \longrightarrow \mathcal{M}$?

## Counterexample

Let $Q_{\text{non}}$ denote the **non-normal** pseudofunctor classifier comonad on **2-Cat**. Then the category of $Q_{\text{non}}$-coalgebras does **not** admit a model structure left-induced from **2-Cat** along $\mathbf{2\text{-}Cat}_{Q_{\text{non}}} \longrightarrow \mathbf{2\text{-}Cat}$.
In $\mathbf{2\text{-}Cat}_{Q_{\text{non}}}$, $\mathbf{1}+\mathbf{1} \longrightarrow \mathbf{1}$ has the RLP wrt all cofibrations, but is not a biequivalence.

Tapadh leibh!