Equivalences of complicial sets ~~§1 Complicial sets~~

A *marked simplicial set* is a simplicial set with a distinguished subset of *marked* or *thin* (or *marked thin*) positive dimensional simplices, ~~such that~~ every ~~non~~degenerate simplex ~~is marked~~ contains

A *morphism* of marked simplicial sets is a morphism of simplicial sets ~~which~~ that preserves marked simplices.

These form a category *msSet*, ~~which is lfp and a quasi-topos (in particular, is cartesian closed~~

$$\begin{tikzcd} \mathrm{msSet} \arrow[r, "U" description] & \mathrm{sSet} \arrow[l, bend right=40, "(-)^\flat"'] \arrow[l, bend left=20, "(-)^\sharp" description] \arrow[r, phantom, ""] \end{tikzcd}$$

$$\mathrm{msSet} \;\underset{\mathrm{core}}{\overset{(-)^\flat}{\;\begin{smallmatrix}\longleftarrow\\ \perp \\ \xrightarrow{\;U\;} \\ \perp \\ \xleftarrow{(-)^\sharp} \\ \perp \\ \longrightarrow\end{smallmatrix}\;}}\; \mathrm{sSet}$$

The functor $U: \mathrm{msSet} \to \mathrm{sSet}$ creates limits & colimits. The category *msSet* is a locally finitely presentable quasi-topos; in particular, it is cartesian closed.

* *msSet* admits a tensored & cotensored simplicial enrichment $X \otimes A := X^\sharp \times A$ ; $\mathrm{Hom}(A,B) := \mathrm{core}(B^A)$.

A morphism $f$ in *msSet* is a monomorphism iff $U(f)$ is a mono in *sSet*.

A mono $f: X \hookrightarrow Y$ in *msSet* is:
- *regular* if a simplex $x$ of $X$ is marked iff $f(x)$ is marked in $Y$.
- *entire* if $U(f)$ is an isomorphism.

~~The k-~~ *Admissible simplices*. Let $n \geq 1$, $0 \leq k \leq n$.

The *k-admissible n-simplex* $\Delta^k[n]$ is the simplicial set $\Delta[n]$, ~~equipped with the m~~ in which a n.d. simplex $[m] \hookrightarrow [n]$ is marked iff its image contains $\{k-1, k, k+1\} \cap [n]$. Equiv, a n.d. simplex is marked iff it is not contained in

$\partial_{k-1}\Delta[n] \cup \partial_k\Delta[n] \cup \partial_{k+1}\Delta[n]$.

<u>Idea</u>: ~~the~~ $0<k<n$: $\Delta^k[n]$ witnesses $\partial_k\Delta[n]$ as the "composite" of $\partial_{k-1}\Delta[n]$ and $\partial_{k+1}\Delta[n]$ along their common face.

- $k=0$ ($k=n$): $\Delta^k[n]$ witnesses an "equivalence" between $\partial_0\Delta[n]$ and $\partial_1\Delta[n]$ ($\partial_{n-1}\Delta[n]$ and $\partial_n\Delta[n]$)

<u>$n=1$.</u>

$$0 \xrightarrow{\sim} 1$$

$$\Delta^0[1] = \Delta^1[1] = \Delta[1]^\#$$

<u>$n=2$.</u>

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, "\sim"'] & {\scriptstyle 12} & 2 \\ & 1 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[rr] \arrow[dr] & {\scriptstyle 12} & 2 \\ & 1 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[rr] \arrow[dr] & {\scriptstyle 12} & 2 \\ & 1 \arrow[ur, "\sim"'] & \end{tikzcd}$$

$$\Delta^0[2] \qquad\qquad \Delta^1[2] \qquad\qquad \Delta^2[2]$$

<u>$n=3$.</u>

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "\sim"'] \arrow[dr] \arrow[dr, phantom, "\Downarrow", shift left=3] \arrow[dr, phantom, "{\scriptstyle 12}"', shift right=3] & 3 \\ 1 \arrow[r] & 2 \arrow[u] \end{tikzcd} \simeq \begin{tikzcd} 0 \arrow[r] \arrow[d, "\sim"'] \arrow[r, phantom, "{\scriptstyle 12}"', shift right=4] & 3 \\ 1 \arrow[r] \arrow[ur] \arrow[ur, phantom, "\Downarrow", shift right=3] & 2 \arrow[u] \end{tikzcd} \qquad \Delta^0[3]$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\Downarrow", shift left=3] \arrow[dr, phantom, "{\scriptstyle 12}"', shift right=3] & 3 \\ 1 \arrow[r] & 2 \arrow[u] \end{tikzcd} \simeq \begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[r, phantom, "\Downarrow"', shift right=4] & 3 \\ 1 \arrow[r] \arrow[ur] \arrow[ur, phantom, "\Downarrow", shift right=3] & 2 \arrow[u] \end{tikzcd} \qquad \Delta^1[3]$$

etc.

- $\Delta^k[n]'$ denotes the entire superset of $\Delta^k[n]$ in which $\partial_{k-1}\Delta[n]$ and $\partial_{k+1}\Delta[n]$ are marked
- $\Delta^k[n]''$ denotes the entire superset of $\Delta^k[n]'$ in which $\partial_k \Delta[n]$ is marked.

<u>Def.</u> A marked simplicial set is a <u>complicial set</u> if it has the RLP wrt:

- the <u>complicial horn extension</u>

$$\Lambda^k[n] \xhookrightarrow[\text{regular}]{} \Delta^k[n] \qquad \forall\, n \geq 1,\ 0 \leq k \leq n$$

- the <u>complicial thinness extension</u>

$$\Delta^k[n]' \xhookrightarrow[\text{entire}]{} \Delta^k[n]'' \qquad \forall\, n \geq 2,\ 0 \leq k \leq n.$$

<u>Examples</u>: 1) Any maximally marked Kan complex.
2) Any naturally-marked quasi-category.
3) The Street nerve of any strict $\omega$-category in which the "identities" are marked.

~~The~~

$A \xrightarrow{f} B$

<u>Def.</u> A morphism of marked simplicial sets is a <u>marked homotopy equivalence</u> if [it is a simplicial homotopy equivalence in the simplicial enrichment of msSet]

$$\exists\ g : B \longrightarrow A, \quad \Delta[1]^{\#} \times A \xrightarrow{\alpha} A, \quad 1_A \simeq gf$$

$$\Delta[1]^{\#} \times B \longrightarrow B, \quad 1_B \simeq fg.$$

<u>Thm</u> (Verity) There exists a simplicial model structure on <u>msSet</u> in which the cofibrations are the monomorphisms and the fibrant objects are the complicial sets. A morphism of complicial sets is a weak equivalence iff it is a marked homotopy equivalence.

~~Prop. msSet~~

~~In this talk, we with~~

<u>Prop</u> (Verity)
$$\begin{tikzcd} \underline{\mathrm{msSet}} \arrow[r, shift right, "\mathrm{core}"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{sSet}} \arrow[l, shift right, "(-)^{\#}"'] \end{tikzcd}$$
is a Quillen adjunction between the model structures for complicial sets & Kan complexes.
In particular, for every complicial set $A$, its core $\mathrm{core}(A)$ is a Kan complex.

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~~Ex~~ (right)
<u>The suspension-hom adjunction</u>

$$\begin{tikzcd} \partial\Delta[1] \backslash \underline{\mathrm{msSet}} \arrow[r, shift right, "\mathrm{Hom}"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{msSet}} \arrow[l, shift right, "\Sigma"'] \end{tikzcd}$$

The (right) <u>suspension</u> $\Sigma(X)$ of a marked simplicial set $X$ is defined by the pushout:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X \star \Delta[0] \arrow[d] \\ \Delta[0] \arrow[r] & \Sigma(X) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

marked join

~~$X \to X \star \Delta[0]$~~

$\Sigma(X)$ has two 0-simplices, and its n.d. (marked) $(n+1)$-simplices are in bijection with the n.d. (marked) $n$-simplices of $X$.

**Examples:** $\Sigma(\Delta[0]) = \bullet \longrightarrow \bullet = \Delta[1]$ | $\Sigma(\partial\Delta[1]) = \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & y \end{tikzcd}$

$$\Sigma(\Delta[1]) = \begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] & y \arrow[dl, phantom, "\Downarrow\alpha"] \\ x \arrow[ur, "g"'] & \end{tikzcd} =: \mathfrak{D}_2$$

$$\Sigma(\mathfrak{D}_1) = \begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] \arrow[dr, equal] & y \arrow[d, phantom, "\Downarrow\alpha"] \\ x \arrow[r, equal] & x \arrow[u, "g"'] \end{tikzcd} \overset{m}{\Longrightarrow} \begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & y \\ x \arrow[r, equal] \arrow[ur] & x \arrow[u, "g"'] \end{tikzcd} := \mathfrak{D}_3$$

($\mathfrak{D}_n \to A$ = an $n$-cell in $A$)

$\mathfrak{D}_0 = \Delta[0]$, $\mathfrak{D}_{n+1} := \Sigma(\mathfrak{D}_n)$. "free-living $n$-cell"

$\partial\mathfrak{D}_0 = \emptyset$, $\partial\mathfrak{D}_{n+1} := \Sigma(\partial\mathfrak{D}_n)$. "boundary of ..."

$$\Sigma(\Delta[2]) = \begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] \arrow[dr, equal] & y \arrow[d, phantom, "\Downarrow\alpha"] \\ x \arrow[r, equal] & x \arrow[u, "h"'] \end{tikzcd} \overset{m}{\Longrightarrow} \begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta\ \Downarrow\gamma"] & y \\ x \arrow[r, equal] \arrow[ur, "g"] & x \arrow[u, "h"'] \arrow[ul, phantom, "\Downarrow\varepsilon"] \end{tikzcd}$$

Let $X \in$ msSet ~~be a marked~~ and $x, y \in X_0$.
Define the (right) hom marked simplicial set $\mathrm{Hom}_X(x,y)$ so that

- a (marked) $n$-simplex of $\mathrm{Hom}_X(x,y)$ is a (marked) $(n+1)$-simplex whose last vertex is $y$ and whose initial face is degenerate on $x$.

**Prop.** The adj$^n$ $\begin{tikzcd} \partial\Delta[1]\backslash \text{msSet} \arrow[r, shift right, "\mathrm{Hom}"'] \arrow[r, phantom, "\perp"] & \text{msSet} \arrow[l, shift right, "\Sigma"'] \end{tikzcd}$
is a Quillen adjunction. ~~wrt the model structure~~ In particular, the homs of a complicial set are complicial sets.

~~Defn. A 0-cell in A is a 0-simplex in A.~~

[Note that the $n$-cells in $\mathrm{Hom}_A(a,b)$ are the $(n+1)$-cells in $A$]

- Define (marked) $n$-cell in a complicial set. 

<u>Thm</u>. A morphism of complicial sets $A \xrightarrow{f} B$ is a marked ~~homotopy~~ equivalence iff for all $n \geq 0$,

$$\mathrm{Hom}(\mathfrak{R}_n, A) \xrightarrow[\mathrm{Hom}(\mathfrak{R}_n, f)]{} \mathrm{Hom}(\mathfrak{R}_n, B)$$

is a homotopy equivalence of Kan complexes.

<u>Proof</u>. See last section of talk. $\square$

<u>Def</u>. A morphism of complicial sets $A \xrightarrow{f} B$ is:

- <u>essentially surjective on 0-cells</u> if: for all $b \in B_0$, $\exists\, a \in A_0$ and a marked 1-simplex $f(a) \xrightarrow{\sim} b$ in $B$
- <u>ess. surj on $n$-cells</u> if, for all $a, b \in A_0$,
  $$\mathrm{Hom}_A(a,b) \xrightarrow{f} \mathrm{Hom}_B(f(a), f(b))$$
  is ess surj on $(n-1)$-cells.
  (ie for every $\partial \mathfrak{D}_n \xrightarrow{(a,b)} A$ parallel pair of $(n-1)$-cells in $A$, and every $f(a) \xrightarrow{\beta} f(b)$ $n$-cell in $B$, $\exists$ $n$-cell $a \xrightarrow{\alpha} b$ in $A$ and a marked $(n+1)$-cell
  $$\begin{tikzcd} f(a) \arrow[r, bend left, "f(\alpha)"] \arrow[r, bend right, "\beta"'] \arrow[r, phantom, "\wr"] & f(b) \end{tikzcd}$$
  in $B$)
- <u>conservative on $n$-cells</u> $(n \geq 1)$ if: for any $n$-cell $a \xrightarrow[u]{} b$ in $A$, if $f(u)$ is ~~marked~~ thin in $B$, then $u$ is thin in $A$.

<u>Prop.</u> Let $A \xrightarrow{f} B$ be a morphism of complicial sets. if $f$ is
(i) ess. surj. on $n$-cells $\forall n \geq 0$, and
(ii) cons. on $n$-cells $\forall n \geq 1$,
then $\mathrm{core}(f) : \mathrm{core}(A) \longrightarrow \mathrm{core}(B)$ is a homotopy equivalence of Kan complexes.

<u>Proof.</u> First show that $\mathrm{core}(f)$ is bijective on $\pi_0$.
- surj: let $b \in B_0$. Since $f$ is ess. surj. on 0-cells, $\exists\, a \in A_0$ and a marked 1-simplex $f(a) \xrightarrow{\sim} b$ in $B$ ~~and in~~ which, being marked, belongs to $\mathrm{core}(B)$.

- inj: let $a, b \in A_0$, and let $f(a) \xrightarrow{\beta} f(b)$ be a 1-simplex in $\mathrm{core}(B)$, i.e. a marked 1-simplex in $B$. Since $f$ is ~~of~~ ess. surj. on 1-cells, $\exists\; a \xrightarrow{\alpha} b$ in $A$ and thin 2-cell

$$\begin{tikzcd} f(a) \arrow[r, bend left, "f(\alpha)"] \arrow[r, bend right, "\beta"'] \arrow[r, phantom, "12"] & f(b) \end{tikzcd} \quad \text{in } B.$$

Since $\beta$ is thin, so is $f(\alpha)$. Since $f$ is cons. on 1-cells, $\alpha$ is thin, and hence belongs to $\mathrm{core}(A)$.

$$\left[\begin{tikzcd} f(a) \arrow[rr, "f(\alpha)", "\sim"'] \arrow[dr, equal, "\sim"'] & \arrow[d, phantom, "12"] & f(b) \\ & f(a) \arrow[ur, "\sim", "\beta"'] & \end{tikzcd}\right]$$

Now, $\forall a \in A_0$ $\Omega^n(\mathrm{core}(A), 1_a) \simeq$ ~~Hom, core~~ $\mathrm{Hom}^n_{\mathrm{core}(A)}(1_a, 1_a)$

$= \mathrm{core}(\mathrm{Hom}^n_A(1_a, 1_a))$,

and $\mathrm{Hom}^n(f)$ has properties (i) & (ii).
So the first part of the proof gives that $f$ is ~~$\pi_n f$~~ bijective on $\pi_n$, $\forall n \geq 1$ and all basepoints.

$\square$

($\Longleftarrow$) Since $\mathrm{Hom}^n(f)$ is also a marked htpy equiv $\forall n \geq 1$ it suffices to show that $f$ is

- ess. surj. on 0-cells &
- cons. on 1-cells.

Let $g : B \to A$, $\alpha : 1_A \xrightarrow{\sim} gf$,

$\beta : 1_B \xrightarrow{\sim} fg$

be ~~the~~ a pseudo-inverse to $f$.

es0: let $b \in B_0$: $\beta_b : b \xrightarrow[u]{\sim} f(g(a))$

cons1: $a \xrightarrow{u} b$ & suppose $f(a) \xrightarrow[\sim]{f(u)} f(b)$.

comp of $\alpha$ at $u$:

$$\begin{tikzcd} a \arrow[r, "u", "\sim"'] \arrow[d, "\alpha a"', "\sim"] \arrow[dr, "\sim", "\alpha_u"'] & b \arrow[d, "\alpha b", "\sim"'] \\ gfa \arrow[r, "\sim", "gfu"'] & gfb \end{tikzcd}$$

$\square$

!! Gotta prove the easy direction first!

**Thm.** ~~Let~~ $A \xrightarrow{f} B$ be ~~a~~ morphism of complicial sets. ~~If f is~~ $A \xrightarrow{f} B$ is a marked htpy equiv iff

(i) ess. surj. on $n$-cells $\forall n \geq 0$, and

(ii) cons. on $n$-cells $\forall n \geq 1$.

~~then f is a marked homotopy equivalence~~.

~~Ex~~ $f : A \to B$ Prove that if $f$ is a ~~map~~ marked htpy equiv, then it's got (i) & (ii).

**Proof.** By previous Thm, suffices to show that

$$\mathrm{Hom}(\mathfrak{D}_n, A) \xrightarrow[\mathrm{Hom}(\mathfrak{D}_n, f)]{} \mathrm{Hom}(\mathfrak{D}_n, B)$$

is an equivalence of Kan complexes for all $n \geq 0$.

We'll prove this by induction on $n$.

The $n=0$ case is precisely the preceding Prop$^n$.

$\underline{n > 0}$. Consider the comm. square of Kan complexes

$$\begin{tikzcd} \mathrm{Hom}(\mathfrak{D}_n, A) \arrow[r, "(1,f)"] \arrow[d] & \mathrm{Hom}(\mathfrak{D}_n, B) \arrow[d] \\ \mathrm{Hom}(\partial\mathfrak{D}_n, A) \arrow[r, "\sim", "(1,f)"'] & \mathrm{Hom}(\partial\mathfrak{D}_n, B) \end{tikzcd} \qquad (*)$$

By the induction hypothesis, and the ~~fact that~~ (htpy) pushout square

$$\begin{tikzcd} \partial\mathfrak{D}_{n-1} \arrow[r] \arrow[d] & \mathfrak{D}_n \arrow[d] \\ \mathfrak{D}_{n-1} \arrow[r] & \partial\mathfrak{D}_{n-1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

)

the bottom map is a htpy equiv.

It therefore suffices to show that $(*)$ is a htpy pullback square. For this, since the vertical maps are Kan fibrations, it suffices to show that, for each parallel pair of $(n-1)$-cells $a, b$ in $A$, the induced map on fibres

$$(**)\ \mathrm{Hom}_{\partial\mathfrak{D}_n}\left(\begin{matrix}\partial\mathfrak{D}_n \\ \downarrow \\ \mathfrak{D}_n\end{matrix}, \begin{matrix}\partial\mathfrak{D}_n \\ \downarrow{\scriptstyle (a,b)} \\ A\end{matrix}\right) \xrightarrow{(1,f)} \mathrm{Hom}_{\partial\mathfrak{D}_n}\left(\begin{matrix}\partial\mathfrak{D}_n \\ \downarrow \\ \mathfrak{D}_n\end{matrix}, \begin{matrix}\partial\mathfrak{D}_n \\ \downarrow{\scriptstyle (fa,fb)} \\ B\end{matrix}\right)$$

is an equivalence of Kan complexes.
But this map is precisely the image of the map

$$\begin{tikzcd} & \partial \mathfrak{D}_n \arrow[dl, "(a,b)"'] \arrow[dr, "(fa, fb)"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd} \qquad (***)$$

under the right adjoint of the Quillen adjunction

$$\partial \mathfrak{D}_n \backslash \underline{msSet} \underset{\mathrm{Hom}_{\mathfrak{D}_n}\left(\begin{smallmatrix}\partial\mathfrak{D}_n \\ \downarrow \\ \mathfrak{D}_n\end{smallmatrix}, -\right)}{\overset{\perp}{\rightleftarrows}} \underline{sSet}$$

whose left adjoint sends $\Delta[0]$ to $\begin{smallmatrix}\partial\mathfrak{D}_n \\ \downarrow \\ \mathfrak{D}_n\end{smallmatrix}$.
But we also have the composite Quillen adjunction

$$\partial \mathfrak{D}_n \backslash \underline{msSet} \underset{\mathrm{Hom}}{\overset{\Sigma}{\rightleftarrows}} \partial \mathfrak{D}_{n-1} \backslash \underline{msSet} \underset{\mathrm{Hom}}{\overset{\Sigma}{\rightleftarrows}} \cdots \underset{\mathrm{Hom}}{\overset{\Sigma}{\rightleftarrows}} \underline{msSet} \underset{\mathrm{core}}{\overset{(-)^\sharp}{\rightleftarrows}} \underline{sSet}$$

whose left adjoint also sends $\Delta[0]$ to $\begin{smallmatrix}\partial\mathfrak{D}_n \\ \downarrow \\ \mathfrak{D}_n\end{smallmatrix}$.
So, by the universal property of the ~~~~ model category for Kan complexes, (**) is an equivalence iff (***) is sent by this ~~right~~ composite right Quillen functor to an equivalence.
But this is just

$$\mathrm{core}(\mathrm{Hom}^n_A(a,b)) \xrightarrow{\mathrm{core}(\mathrm{Hom}^n(f))} \mathrm{core}(\mathrm{Hom}^n_B(fa, fb))$$

which is an equiv by the assumptions on $f$ (which are inherited by $\mathrm{Hom}^n(f)$) and by the preceeding Prop$^n$.

$\square$

~~Def. A complicial set A is 1-saturated~~

**Def.** The *homotopy category* $ho(A)$ of a complicial set A has:

- objects are the 0-simplices of A
- hom-sets $(ho\,A)(a,b) := \pi_0(\mathrm{core}(\mathrm{Hom}_A(a,b)))$,

i.e. morphisms of $ho(A)$ are equivalence classes of 1-simplices in A, where $a \xrightarrow{f} b \simeq a \xrightarrow{g} b$ if $\exists$ marked 2-cell
$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & b \end{tikzcd}.$$

The thin 1-simplices in A ~~Def. A co~~ determine a distinguished subset of ~~the~~ "marked" isomorphisms in $ho(A)$; ~~which we~~

**Def.** A complicial set is *1-saturated* if every isomorphism in $ho(A)$ is marked.

**Def.** A complicial set A is *saturated* if

(i) A is 1-saturated

(ii) $\forall n \geq 1$ $\forall (a,b): \partial\mathbb{D}_n \to A$, $\mathrm{Hom}^n_A(a,b)$ is 1-saturated.

"every equivalence n-cell is marked".

**Prop.** Let $A \xrightarrow{f} B$ be a morphism of complicial sets. Suppose f is ess. surj. on n-cells $\forall n \geq 0$. If A is saturated, then f is cons. on n-cells $\forall n \geq 1$.

**Proof.** By (co?) induction, suffices to show that f is cons. on 1-cells.

$f$ ~~ex~~ ess surj on 1-cells & 2-cells
$\Longrightarrow$ $ho(f) : ho(A) \longrightarrow ho(B)$
is ~~an eq~~ fully faithful, hence conservative.

Let $a \xrightarrow{u} b$ in $A$ & suppose $f(a) \xrightarrow{f(u)} f(b)$ in $B$ is thin. Then $f(u)$ is an iso in $ho(B)$, so $u$ is an iso in $ho(A)$ by conservativity of $ho(f)$. But $A$ is saturated; hence $u$ is marked. □

Cor. A morphism of saturated complicial sets is a marked homotopy equivalence iff it is ess. surj. on $n$-cells $\forall n \geq 0$.

Def. Let $n \geq 0$. An $n$-complicial set is a ~~saturated~~ complicial set in which ~~$\forall m > n$~~ every $m$-simplex, for $m > n$, is marked.

Ex. If $A$ is ~~an~~ $n$-complicial, then $Hom_A(a,b)$ is $(n-1)$-complicial.

Cor. A morphism of $n$-complicial sets is a marked htpy equiv iff it is
- ess. surj. on objects, and
- ~~a marl~~ $\forall a, b \in A_0$,
$$Hom_A(a,b) \xrightarrow{f} Hom_B(fa, fb)$$
is a marked htpy equiv.

Equivalences of complicial sets (part II).

IOU:

<u>Thm.</u> A morphism of complicial sets $A \xrightarrow{f} B$ is a (marked homotopy) equivalence iff

$$\mathrm{Hom}(\mathcal{D}_n, A) \xrightarrow[\mathrm{Hom}(\mathcal{D}_n, f)]{} \mathrm{Hom}(\mathcal{D}_n, B)$$

is an equivalence of Kan complexes $\forall n \geq 0$.

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<u>Def.</u> A class $\mathcal{C}$ of marked simplicial sets is <u>saturated by monomorphisms</u> if:

(a) $\mathcal{C}$ is closed under coproducts,

(b) $\forall$
$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & C \arrow[d, tail] \\ B \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$
, $A, B, C \in \mathcal{C}$ $\Rightarrow D \in \mathcal{C}$

(c) $\forall$ $C_0 \rightarrowtail C_1 \rightarrowtail C_2 \rightarrow \cdots$ $\omega$-sequence of monos

$C_n \in \mathcal{C}$ $\forall n$ $\Rightarrow$ $\operatorname{colim}_n(C_n) \in \mathcal{C}$.

(d) $\mathcal{C}$ is ~~closed~~ stable under retracts.

$\cdot n \geq 0$, $\Delta[n]$ = min. marked $n$-simplex

$\cdot n \geq 1$, $\Delta[n]_t$ = thin $n$-simplex.

<u>Prop.</u> Let $\mathcal{C}$ be a class of marked simplicial sets. Suppose that:

(i) $\mathcal{C}$ is saturated by monos,

(ii) $\forall n \geq 0$, $\Delta[n] \in \mathcal{C}$, & $\forall n \geq 1$, $\Delta[n]_t \in \mathcal{C}$.

Then every marked simplicial set belongs to $\mathcal{C}$.

<u>proof.</u> First, prove by induction on $n \geq 0$ that every $n$-skeletal marked simplicial set belongs to $\mathcal{C}$.

<u>$n=0$.</u> $\quad sk_0(X) \cong \coprod_{X_0} \Delta[0] \in \mathcal{C}$

$\underline{n>0}$. Suppose every $(n-1)$-skeletal marked simplicial set belongs to $\mathcal{C}$.

$$\begin{tikzcd}
\sum\limits_{\substack{\text{unmarked}\\ n\text{-simplices}\\ \text{of } X}} \partial\Delta[n] + \sum\limits_{\substack{\text{marked}\\ \text{n.d.}\\ n\text{-simplices}\\ \text{of } X}} \partial\Delta[n] \arrow[rr] \arrow[d, tail] & & sk_{n-1}(X) \arrow[d] \\
\sum\limits_{\substack{\text{unmarked}\\ n\text{-simplices}\\ \text{of } X}} \Delta[n] + \sum\limits_{\substack{\text{marked}\\ \text{n.d.}\\ n\text{-simplices}\\ \text{of } X}} \Delta[n]_t \arrow[rr] & & \ulcorner sk_n(X) \in \mathcal{C}
\end{tikzcd}$$

Hence, $\forall n \geq 0$, any $n$-skeletal marked simplicial set belongs to $\mathcal{C}$.

Finally, for any marked simplicial set $X$, we have
$$X = \varinjlim \left( sk_0(X) \hookrightarrow sk_1(X) \hookrightarrow \cdots \right) \in \mathcal{C}. \qquad \square$$

<u>Recall</u>. The <u>Verity model structure</u> for complicial sets on msSet:

- cofibrations $\equiv$ monos
- fibrant objects $=$ complicial sets.
- weak equivalence $=:$ weak equivalence.

<u>Def</u>. A class $\mathcal{C}$ of marked simplicial sets is <u>closed under homotopy colimits</u> if

(a) $\mathcal{C}$ is saturated by monos, and

(b) $\forall$ w.e. $X \xrightarrow{\sim} Y$, if one of $X, Y$ belongs to $\mathcal{C}$, so does the other

<u>Recall</u>. the Quillen adjunction

$$\begin{tikzcd} \partial\Delta[1]\backslash \underline{msSet} \arrow[r, shift right, "\mathrm{Hom}"'] \arrow[r, phantom, "\perp"] & \underline{msSet} \arrow[l, shift right, "\Sigma"'] \end{tikzcd}.$$

$$\begin{tikzcd} \Delta[n] \arrow[r, hook, "\delta^{n+1}"] \arrow[d] & \Delta[n+1] \arrow[d] \\ \Delta[0] \arrow[r] & \ulcorner\Sigma(\Delta[n]) \end{tikzcd} \qquad \begin{tikzcd} \Delta[n] \arrow[r] \arrow[d] & \Delta[n+1]_t \arrow[d] \\ \Delta[0] \arrow[r] & \ulcorner\Sigma(\Delta[n]_t) \end{tikzcd}$$

- Globes : $\mathfrak{D}_0 := \Delta[0]$ $\quad (\mathfrak{D}_1)_t = \Delta[1]_t$

$\mathfrak{D}_{n+1} := \Sigma(\mathfrak{D}_n)$ $\quad (\mathfrak{D}_{n+1})_t := \Sigma((\mathfrak{D}_n)_t)$.

$\mathfrak{D}_0 \qquad \mathfrak{D}_1 \qquad \mathfrak{D}_2 \qquad \ldots$

~~Lemma (Verity "minor filling lemma~~

- The $k$-admissible $n$-simplex : $n \geq 1$, $0 \leq k \leq n$

$\Delta^k[n]$ is the simplicial set $\Delta[n]$ ~~with the~~ ~~mar~~ with the marking:

a n.d. simplex $[m] \hookrightarrow [n]$ is thin iff its image contains $\{k-1, k, k+1\} \cap [n]$.

The regular inclusion

$$\Lambda^k[n] \xrightarrow{\sim} \Delta^k[n]$$

is a weak equivalence.

Note: $(0 \leq i \leq n)$

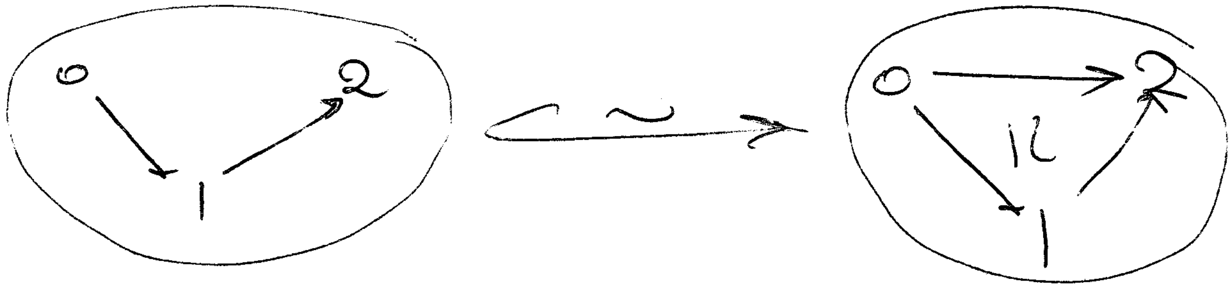
$$\partial_i \Delta^k[n] = \begin{cases} \Delta^{k-1}[n-1] & \text{if } 0 \leq i \leq k-2 \\ \Delta[n-1] & \text{if } k-1 \leq i \leq k+1 \\ \Delta^k[n-1] & \text{if } k+2 \leq i \leq n \end{cases}.$$

<u>Lemma</u>. (Verity – "minor filling lemma").
Let $n \geq 2$, $0 < k < n$.
Then $\partial_{k-1}\Delta^k[n] \cup \partial_{k+1}\Delta^k[n] \xhookrightarrow{\sim} \Delta^k[n]$

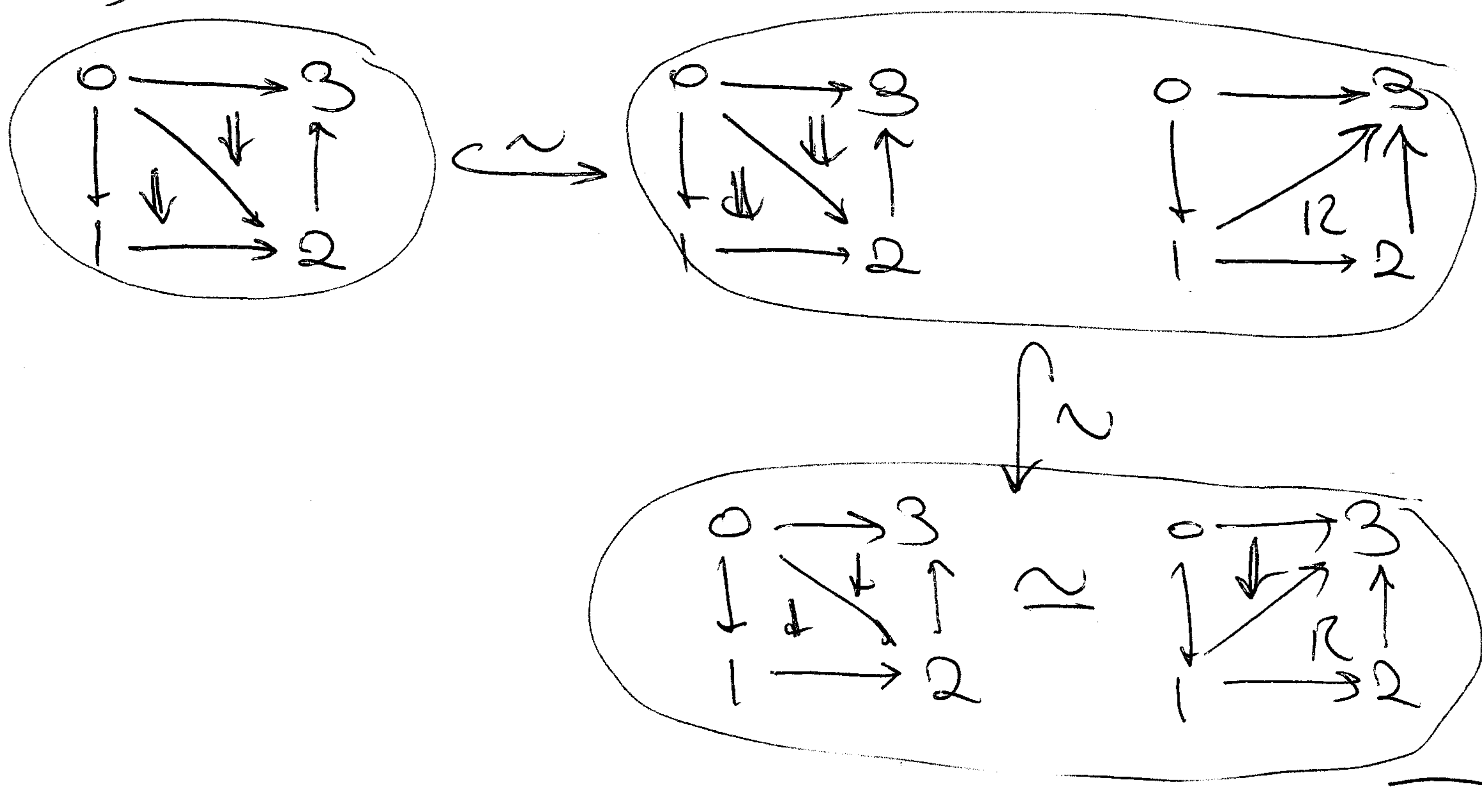
is a weak equivalence.

$$\left(\begin{tikzcd}
\Delta[n-2] \arrow[r, "\delta^{k-1}"] \arrow[d, "\delta^k"'] & \Delta[n-1] \arrow[d] \arrow[ddr, bend left, "\delta^{k+1}"] & \\
\Delta[n-1] \arrow[r] \arrow[drr, bend right, "\delta^{k-1}"'] & \cancel{\partial_{k-1}\Delta^k[n] \cup \partial_{k+1}\Delta^k[n]} \arrow[dr, "\sim"] & \\
 & & \Delta^k[n]
\end{tikzcd}\right)$$

e.g. $n=2, k=1$.



$n=3, k=2$.



<u>Prop</u>. Let $\mathcal{C}$ be a class of marked simplicial sets. Suppose that
(a) $\mathcal{C}$ is closed under homotopy colimits, and
(b) ~~$\mathcal{D}_n$~~ $\forall n \geq 0$, $\mathcal{D}_n \in \mathcal{C}$.
Then every marked simplicial set belongs to $\mathcal{C}$.

<u>Proof</u>. We may suppose that $\mathcal{C}$ is the <u>minimal</u> class with properties (a) & (b).
First, show that $\mathcal{C}$ is closed under suspension.
Let $\mathcal{D} := \Sigma^{-1}(\mathcal{C}) = \{ X \in \text{msSet} \mid \Sigma(X) \in \mathcal{C} \}$.

Then, since $\Sigma$ is a left Quillen functor, $\mathcal{D}$ also has properties (a) and (b).

$$\begin{tikzcd} \coprod_i \partial\Delta[1] \arrow[r] \arrow[d, tail] & \partial\Delta[1] \arrow[d] \\ \coprod_i \Sigma(C_i) \arrow[r] & \Sigma(\coprod_i C_i) \end{tikzcd}$$

Hence, by minimality of $\mathcal{C}$, we have ~~$\mathcal{D} \subseteq \Sigma^{-1}(\mathcal{C})$~~. $\mathcal{C} \subseteq \Sigma^{-1}(\mathcal{C})$.
i.e. $X \in \mathcal{C} \Longrightarrow \Sigma(X) \in \mathcal{C}$.

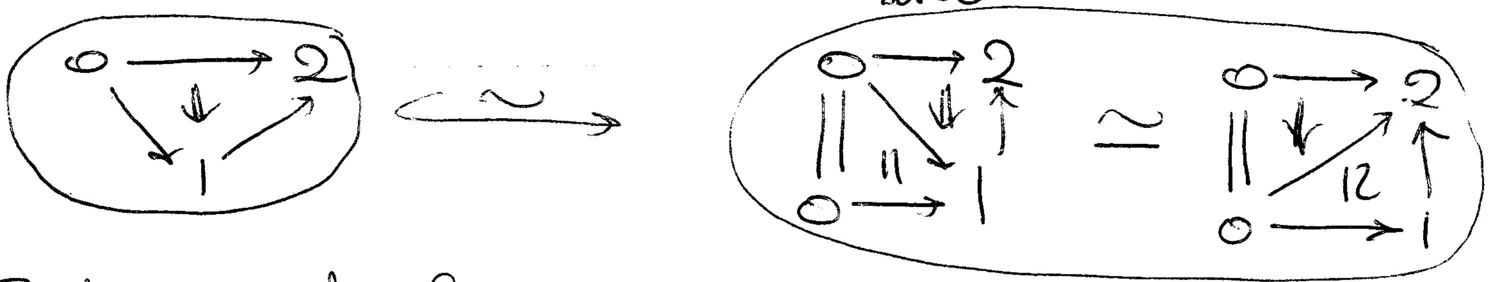
Now, by a previous Prop$^{n}$, it suffices to show that $\forall n \geq 0$, $\Delta[n] \in \mathcal{C}$ & $\forall n \geq 1$, $\Delta[n]_t \in \mathcal{C}$.

Low dim. cases:

- $\Delta[0] = \mathcal{D}_0 \in \mathcal{C}$
- $\Delta[1] = \mathcal{D}_1 \in \mathcal{C}$
- $\Delta[0] \underset{\sim}{\overset{s^1}{\hookrightarrow}} \Delta[1]_t$, $\Delta[0] \in \mathcal{C} \Longrightarrow \Delta[1]_t \in \mathcal{C}$.
  $(*) \Longrightarrow \mathcal{D}_{n,t} \in \mathcal{C}$ $\forall n \geq 1$

We prove by induction on $n \geq 1$ that
$\Delta[n] \in \mathcal{C}$ & $\Delta[n]_t \in \mathcal{C}$.
$n=1$, above.
Let's look at $n=2$ case.

$$\begin{tikzcd} \partial_1\Delta^2[3] \cup \partial_3\Delta^2[3] \arrow[r, "(id,\, s^2\sigma^0)"] \arrow[d, hook, "\sim"] & \Delta[2] \arrow[d, hook, "\sim"] \\ \Delta^2[3] \arrow[r] & \Delta^2[3]/\partial_3 \text{ degen at } 0 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

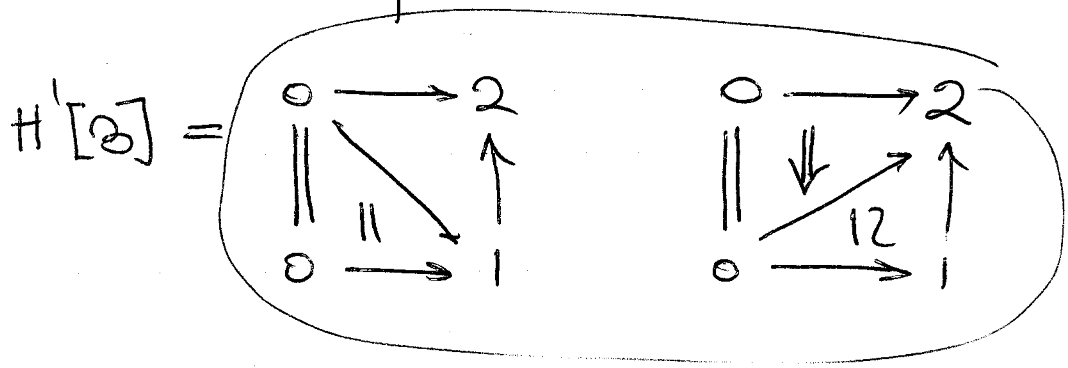


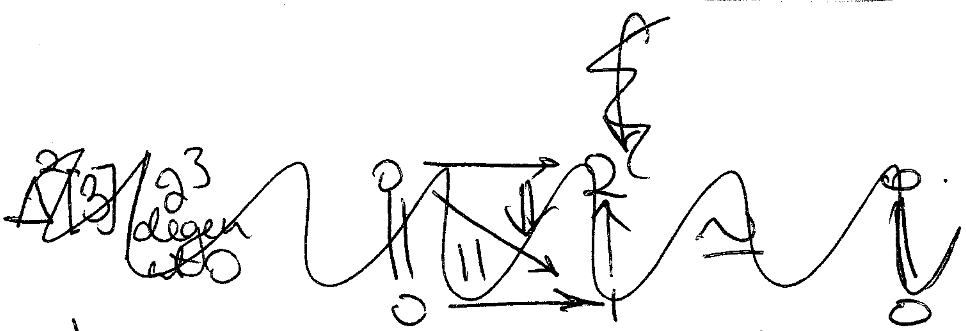
But we also have

$$\begin{tikzcd} \Lambda^1[3] \arrow[r, two heads] \arrow[d, hook, "\sim"'] & H^1[3] \arrow[d, hook, "\sim"] \\ \Delta^1[3] \arrow[r, "\exists"'] & \Delta^2[3]/\partial_3 \text{ degen at } 0 \end{tikzcd}$$

regular image

This is a pushout!





$H^1[3] =$ (degen. 2-simplex) $\cup$ (admissible 2-simplex) $\cup\ \mathcal{I}_2$ .

Now, these belong to $\mathcal{C}$, since $\Delta[1] \in \mathcal{C}$ by induction.

$$\begin{array}{ccc} \Lambda^1[2] & \xhookrightarrow{\sim} & \Delta^1[2] \\ \cap & & \cap \\ \mathcal{C} & \therefore & \mathcal{C} \\ \text{(1-skeleton)} & & \end{array}$$

Hence have

$$\begin{array}{ccccc} \Delta[2] & \xhookrightarrow{\sim} & \cancel{\Delta[3]/\text{degen}} & \xleftarrow{\sim} & H^1[3] \\ \cap & & & & \cap \\ \therefore\ \mathcal{C} & & & & \mathcal{C} \end{array}$$

~~A similar argument shows~~

$$\begin{array}{ccccc} \Lambda^1[2] & \xhookrightarrow{\sim} & \Delta^1[2] & = & \Delta[2]_t \\ \cap & & & & \cap \\ \mathcal{C} & & & \therefore & \mathcal{C} \end{array}$$

$\underline{n>2}$. Note $\Delta[n-1] \in \mathcal{C} \Longrightarrow \Sigma'(\Delta[n-1]) \in \mathcal{C}$.
& $\Sigma'(\Delta[n-1]) = \Delta[n]/\langle 0 \cdots n-1 \rangle$.
Also $\Lambda^k[n] \xhookrightarrow{\sim} \Delta^k[n] \Longrightarrow \Delta^k[n] \in \mathcal{C}$.
We'll show $\forall\ 0 \le k \le n-2$ that
$\Delta[n]/\langle 0 \cdots k \rangle \sim$ a htpy colim of (quotients of) $\Delta[n]/\langle 0 \cdots k+1 \rangle$, admissible $n$-simplices, & degenerate $n$-simplices.

Hence

$\Sigma'(\Delta[n-1]) \in \mathcal{C} \Longrightarrow \Delta[n]/\langle 0 \cdots n-2 \rangle \in \mathcal{C}$

$\| \quad \Longrightarrow \cdots \Longrightarrow \Delta[n] \in \mathcal{C}$.

$\Delta[n]/\langle 0 \cdots \cancel{k} \rangle$

A similar argument, starting with $\Sigma'(\Delta[n-1]_t)$ shows that $\Delta[n]_t \in \mathcal{C}$. $\square$

$$\begin{tikzcd}
\Delta[n] \cup_{\Delta[n-1]} \Delta[n] \arrow[r, "{(\pi, \pi\delta^{k+2}\sigma^{k})}"] \arrow[d, hook, "{(\delta^{k+1}, \delta^{k+3})}"', "\sim"] & \Delta[n]/\langle 0 \cdots k \rangle \arrow[d, hook, "\sim"] \\
\Delta^{k+2}[n+1] \arrow[r] & Q_k \cancel{\Delta^{k+2}}[n+1]
\end{tikzcd}$$

$$\begin{tikzcd}
\Lambda^{k+1}[n+1] \arrow[r, two heads] \arrow[d, hook, "\sim"'] & H^{k+1}[n+1] \arrow[d, hook, "\sim"] \\
\Delta^{k+1}[n+1] \arrow[r] & Q_k[n+1]
\end{tikzcd}$$

<u>Recall</u>. The ~~s~~simplicial hom Quillen bifunctor

$$\underline{msSet}^{op} \times \underline{msSet} \xrightarrow{\mathrm{Hom}} \underline{msSet}$$

$$\mathrm{Hom}(A,B) = \mathrm{core}(B^A).$$

<u>Thm</u>. A morphism of complicial sets ~~is an~~ $A \xrightarrow{f} B$ is an equivalence iff

$$\mathrm{Hom}(Q_n, A) \xrightarrow[\mathrm{Hom}(Q_n, f)]{} \mathrm{Hom}(Q_n, B)$$

is an equivalence of Kan complexes $\forall n \geq 0$.

<u>Proof</u>. ($\Longrightarrow$) $\mathrm{Hom}(X, -) : msSet \longrightarrow sSet$ is a simplicial functor, so preserves simplicial htpy equivs.

($\Longleftarrow$) By Yoneda lemma, it suffices to show that

$$\mathrm{Hom}(X, A) \xrightarrow[\mathrm{Hom}(X,f)]{} \mathrm{Hom}(X, B)$$

is an equivalence $\forall X \in \underline{msSet}$.

Let $\mathcal{C}$ be the class of $X \in \underline{msSet}$ for which $\mathrm{Hom}(X, f)$ is an equivalence.

Since Hom is a right Quillen bifunctor, we have that $\mathcal{C}$ is closed under homotopy colimits.

By assumption, $\mathcal{D}_n \in \mathcal{C}$ $\forall n \geq 0$.

Hence, by the prev. prop$^n$, every marked simplicial set belongs to $\mathcal{C}$. $\square$

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<u>Def</u>. A 1-cell $x \xrightarrow{f} y$ in a complicial set $A$ is an <u>equivalence</u> if $\exists$ $y \xrightarrow{g} x$,

$$\begin{tikzcd} x \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "12"] & x \\ & y \arrow[ur, "g"'] & \end{tikzcd} \quad \& \quad \begin{tikzcd} y \arrow[rr, equal] \arrow[dr, "g"'] & \arrow[d, phantom, "12"] & y \\ & x \arrow[ur, "f"'] & \end{tikzcd}.$$

~~Equivalent~~ i.e. iff $f$ is an iso. in the homotopy category of $A$.

~~Def A complicial s~~

N.B. Every thin 1-cell in a complicial set is an equivalence!

<u>Def</u>. A complicial set is <u>1-saturated</u> if every equivalence 1-cell is thin.

<u>Def</u>. A complicial set $A$ is <u>saturated</u> if
(i) $A$ is <u>1</u>-saturated and
(ii) $\forall a, b \in A_0$, $\mathrm{Hom}_A(a,b)$ is saturated.

That is, $A$ is saturated iff every equivalence $n$-cell is thin $(\forall n \geq 0)$.

Prop. Let $A \xrightarrow{f} B$ be a morphism of complicial sets, and suppose that $A$ is saturated.
If $f$ is ess. surj. on $n$-cells $\forall n \geq 0$, then $f$ is cons. on $n$-cells $\forall n \geq 1$.

Proof. ~~By induction,~~ It suffices to show that $f$ is cons. on 1-cells.
$f$ ess. on 1-cells & 2-cells
$\Rightarrow$ $ho(f) : ho(A) \to ho(B)$ is fully faithful, and hence conservative
~~$\Leftrightarrow \forall u$~~
So for any ~~equiv~~ $a \xrightarrow{u} b$ in $A$,
$f(u)$ thin in $B$
$\Rightarrow$ $f(u)$ an equiv. in $B$
$\Rightarrow$ $u$ an equiv in $A$
$\Rightarrow$ $u$ thin in $A$. $\square$

Cor. A morphism of saturated complicial sets is an equivalence iff it is ess. surj on $n$-cells $\forall n \geq 0$.

Def. Let $n \geq 0$. An $n$-complicial set is a saturated complicial set in which every $m$-simplex, for $m > n$, is thin.

~~Ex.~~ N.B. If $A$ is $n$-complicial, then $Hom_A(a,b)$ is $(n-1)$-complicial $\forall a, b \in A_0$.

Cor. A morphism of $n$-complicial sets is an equivalence iff it is
- ~~it is~~ ~~ess~~ ess. surj. on objects
- ~~is~~ an equivalence on hom $(n-1)$-complicial sets.