A modular proof of the straightening theorem

Alexander Campbell

Centre of Australian Category Theory
Macquarie University

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The straightening theorem

The goal of this talk is to give a new, simple proof of Lurie's (unmarked) Straightening Theorem:

Theorem (Lurie, HTT.2.2.1.2)

For every simplicial set A, the straightening–unstraightening adjunction

$$\begin{tikzcd}[\mathfrak{C}(A)^{\mathrm{op}}, \mathbf{sSet}] \arrow[r, bend right, "Un_A"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet}/A \arrow[l, bend right, "St_A"']\end{tikzcd}$$

is a Quillen equivalence between the projective Kan model structure on $[\mathfrak{C}(A)^{\mathrm{op}}, \mathbf{sSet}]$ and the contravariant model structure on $\mathbf{sSet}/A$.

This theorem is an ∞-categorical generalisation of the classical result that, for any category $\mathcal{C}$, there is an equivalence between the category of presheaves over $\mathcal{C}$ and the category of discrete fibrations over $\mathcal{C}$:

$$\begin{tikzcd}[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \arrow[r, bend right, "\text{category of elements}"'] \arrow[r, phantom, "\sim"] & \mathrm{DFib}/\mathcal{C} \arrow[l, bend right, "\text{presheaf of fibres}"']\end{tikzcd}$$

A modular proof of the straightening theorem

Since the publication of HTT, alternative proofs of the Straightening Theorem have been given by Stevenson (arXiv:1512.04815) and by Heuts–Moerdijk (arXiv:1602.01274).

The proof I am presenting today is based on ideas which may be found in §14 and §51 of Joyal's *Notes on quasi-categories*, and uses my recent proof of Joyal's Cylinder Conjecture (AusCatSem Oct–Nov 2019, arXiv:1911.02631).

The basic idea of this proof is to express the straightening–unstraightening adjunction as the composite of three adjunctions (in fact, two adjunctions and one equivalence)

$$[\mathfrak{C}(A)^{\mathrm{op}}, \mathbf{sSet}] \underset{\longrightarrow}{\overset{\longleftarrow}{\sim}} \mathbf{sSet}\text{-}\mathbf{Cyl}(\mathfrak{C}(A), \mathbf{1}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Cyl}(A, \Delta[0]) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{sSet}/A$$

and to show that each of these adjunctions is a Quillen equivalence (between appropriate model structures).

During this talk, we shall proceed along this diagram from left to right, beginning with the equivalence of categories.

Plan

1. Simplicial profunctors vs simplicial cylinders
2. Simplicial cylinders vs quasi-categorical cylinders
3. Quasi-categorical cylinders vs right fibrations
4. The straightening theorem

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The collage of a simplicial profunctor

Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of simplicial categories (= simplicially-enriched categories). A **simplicial profunctor** $M\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a simplicial functor $M\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \longrightarrow \mathbf{sSet}$.

The **collage** of a simplicial profunctor $M\colon \mathcal{A} \longrightarrow \mathcal{B}$ is the simplicial category $\mathit{Coll}(M)$ with set of objects $\mathrm{ob}\,\mathit{Coll}(M) = \mathrm{ob}\,\mathcal{A} + \mathrm{ob}\,\mathcal{B}$, with simplicial hom-sets

$$\mathit{Coll}(M)(x,y) = \begin{cases} \mathcal{A}(x,y) & \text{if } x,y \in \mathcal{A} \\ \mathcal{B}(x,y) & \text{if } x,y \in \mathcal{B} \\ M(x,y) & \text{if } x \in \mathcal{A},\ y \in \mathcal{B} \\ \emptyset & \text{if } x \in \mathcal{B},\ y \in \mathcal{A}, \end{cases}$$

and with identities and composition defined by those of $\mathcal{A}$ and $\mathcal{B}$ and by the action of M on homs.
We get a cospan of simplicial categories

$$\mathcal{A} \longrightarrow \mathit{Coll}(M) \longleftarrow \mathcal{B}.$$

Following Joyal, we call the cospans of simplicial categories that arise in this way **simplicial cylinders** (alias **two-sided codiscrete cofibrations**) from $\mathcal{A}$ to $\mathcal{B}$.

Simplicial cylinders

More precisely, we say that a cospan of simplicial categories $\mathcal{A} \longrightarrow \mathcal{C} \longleftarrow \mathcal{B}$ is a **simplicial cylinder** from $\mathcal{A}$ to $\mathcal{B}$ if it satisfies the following equivalent conditions.

Lemma

Let $\mathcal{A} \longrightarrow \mathcal{C} \longleftarrow \mathcal{B}$ be a cospan of simplicial categories. TFAE:

1. *This cospan is isomorphic to the cospan $\mathcal{A} \longrightarrow \mathit{Coll}(M) \longleftarrow \mathcal{B}$ for some simplicial profunctor M from $\mathcal{A}$ to $\mathcal{B}$.*
2. *The simplicial functor $\mathcal{A} \longrightarrow \mathcal{C}$ is a sieve inclusion and $\mathcal{B} \longrightarrow \mathcal{C}$ is its complementary cosieve inclusion. That is:*
   - *the induced function $\mathrm{ob}\,\mathcal{A} + \mathrm{ob}\,\mathcal{B} \longrightarrow \mathrm{ob}\,\mathcal{C}$ is a bijection,*
   - *the simplicial functors $\mathcal{A} \longrightarrow \mathcal{C}$ and $\mathcal{B} \longrightarrow \mathcal{C}$ are fully faithful, and*
   - *for each pair of objects $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the simplicial hom-set $\mathcal{C}(b, a)$ is empty.*
3. *There is a simplicial functor $\mathcal{C} \longrightarrow \mathbf{2}$ whose fibres above 0 and 1 are $\mathcal{A}$ and $\mathcal{B}$ respectively:*

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d] & \mathcal{B} \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ \{0\} \arrow[r] & \mathbf{2} & \{1\} \arrow[l] \end{tikzcd}$$

Simplicial profunctors vs simplicial cylinders

Let $\mathbf{sSet\text{-}Cyl}(\mathcal{A},\mathcal{B})$ denote the full subcategory of the cospan category $(\mathcal{A}+\mathcal{B})/\mathbf{sSet\text{-}Cat}$ consisting of the simplicial cylinders from $\mathcal{A}$ to $\mathcal{B}$.

We recall the following classical result from enriched category theory (see Street, Fibrations in bicategories).

Proposition

For each pair of simplicial categories $\mathcal{A}$ and $\mathcal{B}$, the collage construction defines an equivalence of categories

$$[\mathcal{A}^{\mathrm{op}} \times \mathcal{B}, \mathbf{sSet}] \simeq \mathbf{sSet\text{-}Cyl}(\mathcal{A},\mathcal{B})$$

between the category of simplicial profunctors from $\mathcal{A}$ to $\mathcal{B}$ and the category of simplicial cylinders from $\mathcal{A}$ to $\mathcal{B}$.

Note that the pseudo-inverse of the collage construction sends a cylinder

$$\mathcal{A} \xrightarrow{I} \mathcal{C} \xleftarrow{J} \mathcal{B}$$

to the "restricted hom" simplicial functor

$$\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \xrightarrow{I^{\mathrm{op}} \times J} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}} \mathbf{sSet}.$$

A model structure for simplicial profunctors

The model structure for Kan complexes

On the category of simplicial sets, there exists a model structure in which:

- cofibration $\equiv$ monomorphism,
- weak equivalence $\equiv$ weak homotopy equivalence, and
- fibration $\equiv$ Kan fibration.

A simplicial set is fibrant in this model structure iff it is a Kan complex.

The projective Kan model structure for simplicial profunctors

Let $\mathcal{C}$ be a simplicial category (e.g. $\mathcal{C} = \mathcal{A}^{\mathrm{op}} \times \mathcal{B}$). The simplicial functor category $[\mathcal{C}, \mathbf{sSet}]$ admits a model structure in which a morphism $\theta \colon F \to G$ is a **weak equivalence** (resp. **fibration**) iff for each object $c \in \mathcal{C}$, the morphism of simplicial sets $\theta_c \colon Fc \to Gc$ is a **weak homotopy equivalence** (resp. **Kan fibration**).
A simplicial functor $\mathcal{C} \to \mathbf{sSet}$ is fibrant in this model structure iff it takes values in Kan complexes.

A model structure for simplicial cylinders I

Observation (The Dwyer–Kan model structure for simplicial cylinders)

Under the equivalence of categories

$$[\mathcal{A}^{\mathrm{op}} \times \mathcal{B}, \mathbf{sSet}] \simeq \mathbf{sSet\text{-}Cyl}(\mathcal{A}, \mathcal{B}),$$

the projective Kan model structure on $[\mathcal{A}^{\mathrm{op}} \times \mathcal{B}, \mathbf{sSet}]$ corresponds to a model structure on $\mathbf{sSet\text{-}Cyl}(\mathcal{A}, \mathcal{B})$, which we call the **Dwyer–Kan model structure**.

Let $\mathrm{ho}\colon \mathbf{sSet\text{-}Cat} \longrightarrow \mathbf{Cat}$ denote the functor that sends a simplicial category to its **homotopy category**, i.e. its change of base along the functor $\pi_0\colon \mathbf{sSet} \longrightarrow \mathbf{Set}$.

Definition (Dwyer–Kan equivalences and isofibrations)

A simplicial functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is said to be a **Dwyer–Kan equivalence** (resp. **Dwyer–Kan isofibration**) if

- the functor $\mathrm{ho}(F)\colon\ \mathrm{ho}\,\mathcal{C} \longrightarrow \mathrm{ho}\,\mathcal{D}$ is essentially surjective on objects (resp. an isofibration), and
- for each pair of objects $x, y \in \mathcal{C}$, the morphism of simplicial sets $F\colon \mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ is a weak homotopy equivalence (resp. Kan fibration).

A model structure for simplicial cylinders II

Observe that there is a forgetful functor **sSet-Cyl**$(\mathcal{A}, \mathcal{B}) \longrightarrow$ **sSet-Cat** which sends a simplicial cylinder $\mathcal{A} \longrightarrow \mathcal{C} \longleftarrow \mathcal{B}$ to the simplicial category $\mathcal{C}$. We may use this functor to give another description of the Dwyer–Kan model structure on **sSet-Cyl**$(\mathcal{A}, \mathcal{B})$ in terms of a model structure on **sSet-Cat**.

The Bergner model structure for simplicial categories

The category **sSet-Cat** of simplicial categories admits a model structure in which

- weak equivalence $\equiv$ Dwyer–Kan equivalence, and
- fibration $\equiv$ Dwyer–Kan isofibration.

A simplicial category $\mathcal{C}$ is **fibrant** in this model structure iff each of its simplicial hom-sets $\mathcal{C}(x, y)$ is a Kan complex, i.e. iff $\mathcal{C}$ is a **Kan-enriched category**. Later, we shall use the fact (proved by Lurie) that this model structure is **left proper**, i.e. that the pushout of any weak equivalence along a cofibration is a weak equivalence.

Proposition

The forgetful functor **sSet-Cyl**$(\mathcal{A}, \mathcal{B}) \longrightarrow$ **sSet-Cat** *creates the Dwyer–Kan model structure on* **sSet-Cyl**$(\mathcal{A}, \mathcal{B})$ *from the Bergner model structure on* **sSet-Cat**.

A model structure for simplicial cylinders III

Proposition

The forgetful functor $\mathbf{sSet\text{-}Cyl}(\mathcal{A},\mathcal{B}) \longrightarrow \mathbf{sSet\text{-}Cat}$ *creates the Dwyer–Kan model structure on* $\mathbf{sSet\text{-}Cyl}(\mathcal{A},\mathcal{B})$ *from the Bergner model structure on* $\mathbf{sSet\text{-}Cat}$*.*

Proof.

It is straightforward to show that the forgetful functor creates weak equivalences and fibrations. (Use that any morphism of simplicial cylinders in $\mathbf{sSet\text{-}Cyl}(\mathcal{A},\mathcal{B})$ is bijective on objects and an isofibration on homotopy categories.)

To see that the forgetful functor creates cofibrations, use that the two adjunctions

$$\begin{tikzcd} \mathbf{sSet\text{-}Cyl}(\mathcal{A},\mathcal{B}) \arrow[r, shift left=2, "\text{forget}"] \arrow[r, phantom, "\perp"] & (\mathcal{A}+\mathcal{B})/\mathbf{sSet\text{-}Cat} \arrow[l, shift left=2, "\text{coreflect}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{sSet\text{-}Cyl}(\mathcal{A},\mathcal{B}) \arrow[r, shift right=2, "\text{forget}"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet\text{-}Cat}/(\mathcal{A}\star\mathcal{B}) \arrow[l, shift right=2, "\text{reflect}"'] \end{tikzcd}$$

are Quillen adjunctions, which is not difficult to show. $\square$

Change of base for simplicial cylinders

For each pair of simplicial functors $F\colon \mathcal{A} \longrightarrow \mathcal{C}$ and $G\colon \mathcal{B} \longrightarrow \mathcal{D}$, there is an adjunction (displayed below left)

$$\mathbf{sSet}\text{-}\mathbf{Cyl}(\mathcal{A},\mathcal{B}) \underset{(F,G)^*}{\overset{(F,G)_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{sSet}\text{-}\mathbf{Cyl}(\mathcal{C},\mathcal{D})$$

$$\begin{tikzcd} \mathcal{A}+\mathcal{B} \arrow[r, "F+G"] \arrow[d] & \mathcal{C}+\mathcal{D} \arrow[d] \\ \mathcal{X} \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & (F,G)_!(\mathcal{X}) \end{tikzcd}$$

whose left adjoint sends a simplicial cylinder $\mathcal{A} \longrightarrow \mathcal{X} \longleftarrow \mathcal{B}$ to the pushout simplicial cylinder displayed on the right above.

Proposition

Let $F\colon \mathcal{A} \longrightarrow \mathcal{C}$ and $G\colon \mathcal{B} \longrightarrow \mathcal{D}$ be a pair of DK-equivalences. Then the adjunction $(F,G)_! \dashv (F,G)^*$ displayed above is a Quillen equivalence between the Dwyer–Kan model structures on $\mathbf{sSet}\text{-}\mathbf{Cyl}(\mathcal{A},\mathcal{B})$ *and* $\mathbf{sSet}\text{-}\mathbf{Cyl}(\mathcal{C},\mathcal{D})$.

Proof.

Under the equivalence between simplicial profunctors and simplicial cylinders, this adjunction corresponds to the left Kan extension–restriction adjunction $Lan_{F^{\mathrm{op}}\times G} \dashv res_{F^{\mathrm{op}}\times G}$, which is a Quillen equivalence between the projective Kan model structures since F and G are DK-equivalences, by a result of Dwyer–Kan. □

Plan

Simplicial categories vs quasi-categories

The Joyal model structure for quasi-categories

There exists a unique model structure on the category **sSet** whose cofibrations are the monomorphisms and whose fibrant objects are the quasi-categories. The weak equivalences in this model structure are called **weak categorical equivalences**.

We shall use the following fundamental theorem, which was proved by Lurie in HTT as a consequence of the straightening theorem. Alternative proofs, which do not rely on the straightening theorem, have been given by Joyal and by Dugger–Spivak.

Theorem

The adjunction

$$\begin{tikzcd} \mathbf{sSet}\text{-}\mathbf{Cat} \arrow[r, bend right, "N"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet} \arrow[l, bend right, "\mathfrak{C}"'] \end{tikzcd},$$

whose left adjoint sends a simplicial set to its ***homotopy coherent realisation****, and whose right adjoint sends a simplicial category to its* ***homotopy coherent nerve****, is a Quillen equivalence between the Bergner model structure on* **sSet-Cat** *and the Joyal model structure on* **sSet**.

Quasi-categorical cylinders

Let A and B be a pair of simplicial sets. We say that a cospan of simplicial sets from A to B is a (quasi-categorical) **cylinder** from A to B if it satisfies the following equivalent conditions.

Lemma

Let $i\colon A \longrightarrow C \longleftarrow B\colon j$ be a cospan of simplicial sets. TFAE:

1. *$i\colon A \longrightarrow C$ is a sieve inclusion and $j\colon B \longrightarrow C$ is its complementary cosieve inclusion. That is:*
   - *the function $(i_0, j_0)\colon A_0 + B_0 \longrightarrow C_0$ is a bijection,*
   - *the morphisms $i\colon A \longrightarrow C$ and $j\colon B \longrightarrow C$ are "fully faithful" (i.e. cartesian w.r.t. the functor $(-)_0\colon \mathbf{sSet} \longrightarrow \mathbf{Set}$), and*
   - *there is no 1-simplex in C from a 0-simplex in the image of j to a 0-simplex in the image of i.*
2. *There is a morphism of simplicial sets $C \longrightarrow \Delta[1]$ whose fibres above 0 and 1 are A and B respectively:*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] & B \arrow[l, "j"'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\ \{0\} \arrow[r] & \Delta[1] & \{1\} \arrow[l] \end{tikzcd}$$

Change of base for quasi-categorical cylinders

For each pair of simplicial sets A and B, let $\mathbf{Cyl}(A, B)$ denote the full subcategory of the cospan category $(A + B)/\mathbf{sSet}$ consisting of the cylinders from A to B. For each pair of morphisms of simplicial sets $u \colon A \longrightarrow C$ and $v \colon B \longrightarrow D$, there is an adjunction

$$\begin{tikzcd} \mathbf{Cyl}(A, B) \arrow[r, shift left=2, "(u,v)_!"] \arrow[r, phantom, "\perp"] & \mathbf{Cyl}(C, D) \arrow[l, shift left=2, "(u,v)^*"] \end{tikzcd}$$

whose left adjoint sends a cylinder $A \longrightarrow X \longleftarrow B$ to the pushout cylinder $(u, v)_!(X)$ displayed on the left below,

$$\begin{tikzcd} A + B \arrow[r, "u+v"] \arrow[d] & C + D \arrow[d] \\ X \arrow[r] & (u, v)_!(X) \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd} \qquad \begin{tikzcd} (u, v)^*(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d] \\ A \star B \arrow[r, "u \star v"'] & C \star D \end{tikzcd}$$

and whose right adjoint sends a cylinder $C \longrightarrow Y \longleftarrow D$ to the pullback cylinder $(u, v)^*(Y)$ displayed on the right above.

The Joyal model structure for quasi-categorical cylinders

Proposition

For each pair of simplicial sets A and B, the category $\mathbf{Cyl}(A, B)$ admits a model structure which is created by the forgetful functor $\mathbf{Cyl}(A, B) \longrightarrow \mathbf{sSet}$ from the Joyal model structure for quasi-categories.

This model structure – which we call the **Joyal model structure** on $\mathbf{Cyl}(A, B)$ – was the subject of my recent talks and paper on Joyal's Cylinder Conjecture.
We shall use the following result, which is a consequence of my proof of Joyal's Cylinder Conjecture.

Theorem (arXiv:1911.02631, Theorems 4.7 and 5.4)

For each pair of weak categorical equivalences $u \colon A \longrightarrow C$ and $v \colon B \longrightarrow D$, the change-of-base adjunction

$$\begin{tikzcd} \mathbf{Cyl}(A, B) \arrow[rr, shift left=2, "(u,v)_!"] \arrow[rr, phantom, "\perp"] & & \mathbf{Cyl}(C, D) \arrow[ll, shift left=2, "(u,v)^*"] \end{tikzcd}$$

is a Quillen equivalence between the Joyal model structures on $\mathbf{Cyl}(A, B)$ and $\mathbf{Cyl}(C, D)$. □

Lemma

1. *If the cospan of simplicial sets* $A \longrightarrow C \longleftarrow B$ *is a cylinder, then the cospan of simplicial categories* $\mathfrak{C}A \longrightarrow \mathfrak{C}C \longleftarrow \mathfrak{C}B$ *is a simplicial cylinder.*
2. *If the cospan of simplicial categories* $\mathcal{A} \longrightarrow \mathcal{C} \longleftarrow \mathcal{B}$ *is a simplicial cylinder, then the cospan of simplicial sets* $N\mathcal{A} \longrightarrow N\mathcal{C} \longleftarrow N\mathcal{B}$ *is a cylinder.*

Proof.

For (1), it suffices by cocontinuity to show that, for each pair of integers $m, n \geq 0$, the cospan of simplicial categories $\mathfrak{C}(\Delta[m]) \to \mathfrak{C}(\Delta[m] \star \Delta[n]) \leftarrow \mathfrak{C}(\Delta[n])$ is a simplicial cylinder, which is immediate from the definition of $\mathfrak{C}$ of a simplex.
For (2), use the characterisations of cylinders as pullbacks of the cospan $\{0\} \longrightarrow \mathbf{2} \longleftarrow \{1\}$ (and its nerve), and the fact that N preserves pullbacks. $\square$

Hence we have functors

$$\mathbf{Cyl}(A, B) \xrightarrow{\mathfrak{C}} \mathbf{sSet\text{-}Cyl}(\mathfrak{C}A, \mathfrak{C}B) \qquad \mathbf{sSet\text{-}Cyl}(\mathcal{A}, \mathcal{B}) \xrightarrow{N} \mathbf{Cyl}(N\mathcal{A}, N\mathcal{B})$$

for each pair of simplicial sets A and B, and each pair of simplicial categories $\mathcal{A}$ and $\mathcal{B}$.

Two adjunctions

For each pair of simplicial categories $\mathcal{A}$ and $\mathcal{B}$, there is an adjunction

$$\begin{tikzcd} \textbf{sSet-Cyl}(\mathcal{A},\mathcal{B}) \arrow[r, phantom, "\perp"] \arrow[r, bend right, "N"'] & \textbf{Cyl}(N\mathcal{A},N\mathcal{B}), \arrow[l, bend right, "(\varepsilon_{\mathcal{A}},\varepsilon_{\mathcal{B}})_! \circ \mathfrak{C}"'] \end{tikzcd}$$

whose left adjoint is the composite functor

$$\textbf{Cyl}(N\mathcal{A},N\mathcal{B}) \xrightarrow{\mathfrak{C}} \textbf{sSet-Cyl}(\mathfrak{C}N\mathcal{A},\mathfrak{C}N\mathcal{B}) \xrightarrow{(\varepsilon_{\mathcal{A}},\varepsilon_{\mathcal{B}})_!} \textbf{sSet-Cyl}(\mathcal{A},\mathcal{B}),$$

where $\varepsilon_{\mathcal{A}} \colon \mathfrak{C}N\mathcal{A} \longrightarrow \mathcal{A}$ denotes the counit of the adjunction $\mathfrak{C} \dashv N \colon \textbf{sSet-Cat} \longrightarrow \textbf{sSet}$.

For each pair of simplicial sets A and B, there is an adjunction

$$\begin{tikzcd} \textbf{sSet-Cyl}(\mathfrak{C}(A),\mathfrak{C}(B)) \arrow[r, phantom, "\perp"] \arrow[r, bend right, "(\eta_A,\eta_B)^* \circ N"'] & \textbf{Cyl}(A,B), \arrow[l, bend right, "\mathfrak{C}"'] \end{tikzcd}$$

whose right adjoint is the composite functor

$$\textbf{sSet-Cyl}(\mathfrak{C}A,\mathfrak{C}B) \xrightarrow{N} \textbf{Cyl}(N\mathfrak{C}A,N\mathfrak{C}B) \xrightarrow{(\eta_A,\eta_B)^*} \textbf{Cyl}(A,B),$$

where $\eta_A \colon A \longrightarrow N\mathfrak{C}A$ denotes the unit of the adjunction $\mathfrak{C} \dashv N \colon \textbf{sSet-Cat} \longrightarrow \textbf{sSet}$.

Simplicial cylinders vs quasi-categorical cylinders I

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of Kan-enriched categories. Then the adjunction

$$\begin{tikzcd} \mathbf{sSet}\text{-}\mathbf{Cyl}(\mathcal{A},\mathcal{B}) \arrow[r, shift right=1.5ex, "N"'] \arrow[r, phantom, "\perp"] & \mathbf{Cyl}(N\mathcal{A},N\mathcal{B}) \arrow[l, shift right=1.5ex, "(\varepsilon_{\mathcal{A}},\varepsilon_{\mathcal{B}})_! \circ \mathfrak{C}"'] \end{tikzcd}$$

is a Quillen equivalence between the Dwyer–Kan model structure on $\mathbf{sSet}\text{-}\mathbf{Cyl}(\mathcal{A},\mathcal{B})$ *and the Joyal model structure on* $\mathbf{Cyl}(N\mathcal{A},N\mathcal{B})$.

Proof.

This adjunction is a Quillen adjunction since the functor $N \colon \mathbf{sSet}\text{-}\mathbf{Cat} \longrightarrow \mathbf{sSet}$ is right Quillen w.r.t. the Bergner and Joyal model structures, from which the model structures of the theorem are created by the relevant forgetful functors.

To see that this adjunction is moreover a Quillen equivalence, we show that:

- the left adjoint reflects weak equivalences, and
- the counit morphism for each fibrant object of $\mathbf{sSet}\text{-}\mathbf{Cyl}(\mathcal{A},\mathcal{B})$ is a weak equivalence.

(Continued on next slide...)

Simplicial cylinders vs quasi-categorical cylinders II

N.B. We use that the adjunction $\mathfrak{C} \dashv N \colon \textbf{sSet-Cat} \to \textbf{sSet}$ is a Quillen equivalence.

Proof.

(...continued from previous slide.) Let $f \colon X \longrightarrow Y$ be a morphism in $\textbf{Cyl}(N\mathcal{A}, N\mathcal{B})$, and suppose that $(\varepsilon_{\mathcal{A}}, \varepsilon_{\mathcal{B}})_! \mathfrak{C}(f)$ is a DK-equivalence. In the diagram on the left below, the vertical morphisms are DK-equivalences, since $\mathcal{A}$ and $\mathcal{B}$ are locally Kan and the Bergner model structure is left proper. Hence $\mathfrak{C}(f)$ is a DK-equivalence by the 2-out-of-3 property, and so f is a weak categorical equivalence.

$$
\begin{tikzcd}
\mathfrak{C}N\mathcal{A} + \mathfrak{C}N\mathcal{B} \arrow[r, tail] \arrow[d, "\varepsilon_{\mathcal{A}} + \varepsilon_{\mathcal{B}}"', "\sim"] & \mathfrak{C}X \arrow[r, "\mathfrak{C}f"] \arrow[d, "\sim"] \arrow[dl, phantom, "\ulcorner", very near end] & \mathfrak{C}Y \arrow[d, "\sim"] \arrow[dl, phantom, "\ulcorner", very near end] \\
\mathcal{A} + \mathcal{B} \arrow[r] & \varepsilon_!\mathfrak{C}X \arrow[r, "\sim", "\varepsilon_!\mathfrak{C}f"'] & \varepsilon_!\mathfrak{C}Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{C}N\mathcal{A} + \mathfrak{C}N\mathcal{B} \arrow[r, tail] \arrow[d, "\varepsilon_{\mathcal{A}} + \varepsilon_{\mathcal{B}}"', "\sim"] & \mathfrak{C}N\mathcal{C} \arrow[d, "\sim"] \arrow[dr, bend left, "\sim", "\varepsilon_{\mathcal{C}}"'] \arrow[dl, phantom, "\ulcorner", very near end] & \\
\mathcal{A} + \mathcal{B} \arrow[r] \arrow[rr, bend right] & \varepsilon_!\mathfrak{C}N\mathcal{C} \arrow[r, dotted] & \mathcal{C}
\end{tikzcd}
$$

Now suppose that $\mathcal{A} \longrightarrow \mathcal{C} \longleftarrow \mathcal{B}$ is a fibrant object of $\textbf{sSet-Cyl}(\mathcal{A}, \mathcal{B})$, and hence that $\mathcal{C}$ is locally Kan. As before, the vertical morphisms in the diagram on the right above are DK-equivalences. Hence the counit morphism of our Quillen adjunction at the fibrant simplicial cylinder $\mathcal{C}$, which is the pushout-corner map in the above right diagram, is a DK-equivalence by the 2-out-of-3 property. $\square$

Simplicial cylinders vs quasi-categorical cylinders III

Using the previous theorem, we may show that the other kind of adjunction relating simplicial cylinders and quasi-categorical cylinders is always a Quillen equivalence.

Theorem

For each pair of simplicial sets A and B, the adjunction

$$\mathbf{sSet}\text{-}\mathbf{Cyl}(\mathfrak{C}A, \mathfrak{C}B) \underset{(\eta_A,\eta_B)^* \circ N}{\overset{\mathfrak{C}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Cyl}(A, B),$$

is a Quillen equivalence between the Dwyer–Kan model structure on $\mathbf{sSet}\text{-}\mathbf{Cyl}(\mathfrak{C}A, \mathfrak{C}B)$ *and the Joyal model structure on* $\mathbf{Cyl}(A, B)$.

Proof.

This adjunction is a Quillen adjunction since $\mathfrak{C} \colon \mathbf{sSet} \longrightarrow \mathbf{sSet}\text{-}\mathbf{Cat}$ is left Quillen w.r.t. the Joyal and Bergner model structures, from which the model structures of the theorem are created by the relevant forgetful functors.
(Continued on next slide...)

Simplicial cylinders vs quasi-categorical cylinders IV

Proof.

To see that this Quillen adjunction is moreover a Quillen equivalence, let $F\colon \mathfrak{C}A \longrightarrow \mathcal{C}$ and $G\colon \mathfrak{C}B \longrightarrow \mathcal{D}$ be fibrant replacements in the Bergner model structure, and let $u\colon A \longrightarrow N\mathcal{C}$ and $v\colon B \longrightarrow N\mathcal{D}$ be their transposes under the Quillen equivalence $\mathfrak{C} \dashv N\colon \mathbf{sSet\text{-}Cat} \longrightarrow \mathbf{sSet}$, which are therefore weak categorical equivalences.

It is not difficult to show that the following square of left adjoints commutes up to natural isomorphism.

$$\begin{tikzcd} \mathbf{sSet\text{-}Cyl}(\mathfrak{C}A, \mathfrak{C}B) \arrow[d, "(F,G)_!"', "\sim"] & \mathbf{Cyl}(A,B) \arrow[l, "\mathfrak{C}"'] \arrow[d, "(u,v)_!", "\sim"'] \\ \mathbf{sSet\text{-}Cyl}(\mathcal{C},\mathcal{D}) & \mathbf{Cyl}(N\mathcal{C}, N\mathcal{D}) \arrow[l, "\sim"', "(\varepsilon_{\mathcal{C}},\varepsilon_{\mathcal{D}})_! \circ \mathfrak{C}"] \end{tikzcd}$$

In this diagram, the vertical functors are left Quillen equivalences by our earlier results on change of base of cylinders along DK-equivalences and along weak categorical equivalences. The bottom functor is a left Quillen equivalence by the previous theorem, since $\mathcal{C}$ and $\mathcal{D}$ are locally Kan. Hence the top functor is a left Quillen equivalence, by the 2-out-of-three property for Quillen equivalences. □

Simplicial profunctors vs quasi-categorical cylinders

We have now proceeded 2/3 of the way along our factorisation of the straightening–unstraightening adjunction.

Theorem

For every pair of simplicial sets A and B, the composite adjunction

$$\begin{tikzcd}[\mathfrak{C}(A)^{\mathrm{op}} \times \mathfrak{C}(B), \mathbf{sSet}] \arrow[r, shift right, "\mathit{collage}"'] \arrow[r, phantom, "\sim"] & \mathbf{sSet}\text{-}\mathbf{Cyl}(\mathfrak{C}A, \mathfrak{C}B) \arrow[l, shift right, "\mathit{restricted\ hom}"'] \arrow[r, shift right, "(\eta_A, \eta_B)^* \circ N"'] \arrow[r, phantom, "\perp"] & \mathbf{Cyl}(A, B) \arrow[l, shift right, "\mathfrak{C}"']\end{tikzcd}$$

is a Quillen equivalence between the projective Kan model structure on $[\mathfrak{C}(A)^{\mathrm{op}} \times \mathfrak{C}(B), \mathbf{sSet}]$ *and the Joyal model structure on* $\mathbf{Cyl}(A, B)$*.*

Proof.

This adjunction is the composite of two Quillen equivalences, and is therefore a Quillen equivalence. □

Plan

A cocomma–comma adjunction

(For the remainder of the talk, we let $B = \Delta[0]$.)

For every simplicial set A, there is an adjunction

$$\begin{tikzcd} \mathbf{Cyl}(A, \Delta[0]) \arrow[r, phantom, "\perp"] \arrow[r, bend right, "A/(-)"'] & \mathbf{sSet}/A \arrow[l, bend right, "C^{\triangleright}"'] \end{tikzcd}$$

whose left adjoint sends a simplicial set $p \colon X \longrightarrow A$ over A to the cylinder from A to $\Delta[0]$ defined by the pushout on the left below,

$$\begin{tikzcd} & \Delta[0] \arrow[d] \\ X \arrow[r] \arrow[d, "p"'] & X \star \Delta[0] \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C^{\triangleright}(X, p) \end{tikzcd} \qquad \begin{tikzcd} A/c \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C/c \arrow[d, "\mathrm{dom}"] \\ A \arrow[r, "i"'] & C \end{tikzcd}$$

and whose right adjoint sends a cylinder $i \colon A \longrightarrow C \longleftarrow \Delta[0] \colon c$ to the simplicial set over A defined by the pullback on the right above.

Quasi-categorical cylinders vs right fibrations I

The contravariant model structure

For every simplicial set A, there exists a unique model structure on the slice category $\mathbf{sSet}/A$ whose cofibrations are the monomorphisms and whose fibrant objects are the right fibrations over A.

The following result is a consequence of my proof of Joyal's Cylinder Conjecture.

Theorem (Dual of arXiv:1911.02631, Theorem 6.4)

For every simplicial set A, the adjunction

$$\begin{tikzcd} \mathbf{Cyl}(A, \Delta[0]) \arrow[r, shift right=1ex, "A/(-)"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet}/A \arrow[l, shift right=1ex, "C^{\triangleright}"'] \end{tikzcd}$$

is a Quillen equivalence between the Joyal model structure on $\mathbf{Cyl}(A, \Delta[0])$ *and the contravariant model structure on* $\mathbf{sSet}/A$*.*

Quasi-categorical cylinders vs right fibrations II

Proof.

There is an equivalence of categories $\mathbf{Cyl}(A, \Delta[0]) \simeq [\Delta^{\mathrm{op}}, \mathbf{sSet}/A]$, under which the Joyal model structure on $\mathbf{Cyl}(A, \Delta[0])$ corresponds to the "canonical" model structure on $[\Delta^{\mathrm{op}}, \mathbf{sSet}/A]$ w.r.t. the contravariant model structure (i.e. the Bousfield localisation of the Reedy model structure w.r.t. the contravariant model structure whose local objects are the weakly constant simplicial objects), and under which the adjunction of the theorem corresponds to the adjunction

$$\begin{tikzcd} {[\Delta^{\mathrm{op}}, \mathbf{sSet}/A]} \arrow[r, shift right=1.5ex, "\mathrm{ev}_0"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet}/A \arrow[l, shift right=1.5ex, "\text{constant}"'] \end{tikzcd}$$

which is a Quillen equivalence by a general result of Rezk–Schwede–Shipley. $\square$

Plan

A modular proof of the straightening theorem

Let A be a simplicial set. It is immediate from Lurie's definition of the straightening functor St_A that it is naturally isomorphic to the left adjoint of the composite adjunction

$$[\mathfrak{C}(A)^{\mathrm{op}}, \mathbf{sSet}] \underset{\text{collage}}{\overset{\text{restricted hom}}{\underset{\sim}{\leftrightarrows}}} \mathbf{sSet}\text{-}\mathbf{Cyl}(\mathfrak{C}A, \mathbf{1}) \underset{(\eta_A,\mathrm{id})^* \circ N}{\overset{\mathfrak{C}}{\underset{\perp}{\leftrightarrows}}} \mathbf{Cyl}(A, \Delta[0]) \underset{A/(-)}{\overset{C^{\triangleright}}{\underset{\perp}{\leftrightarrows}}} \mathbf{sSet}/A.$$

In the three preceeding sections, we have shown these three adjunctions to be Quillen equivalences between the projective Kan model structure on $[\mathfrak{C}(A)^{\mathrm{op}}, \mathbf{sSet}]$, the Dwyer–Kan model structure on $\mathbf{sSet}\text{-}\mathbf{Cyl}(\mathfrak{C}A, \mathbf{1})$, the Joyal model structure on $\mathbf{Cyl}(A, \Delta[0])$, and the contravariant model structure on $\mathbf{sSet}/A$.
Hence we may deduce the following theorem.

Theorem (Lurie)

For every simplicial set A, the straightening–unstraightening adjunction

$$[\mathfrak{C}(A)^{\mathrm{op}}, \mathbf{sSet}] \underset{Un_A}{\overset{St_A}{\underset{\perp}{\leftrightarrows}}} \mathbf{sSet}/A$$

is a Quillen equivalence between the projective Kan model structure on $[\mathfrak{C}(A)^{\mathrm{op}}, \mathbf{sSet}]$ and the contravariant model structure on $\mathbf{sSet}/A$. □