On weighted homotopy limits in Kan-enriched categories

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Weighted limits as conical limits

This talk concerns a generalisation of the following classical result.

Proposition

Let $W\colon \mathcal{A} \longrightarrow \mathbf{Set}$ *and* $F\colon \mathcal{A} \longrightarrow \mathcal{C}$ *be functors. Then we have*

$$\{W, F\} \cong \lim\left(\mathrm{el}(W) \xrightarrow{P} \mathcal{A} \xrightarrow{F} \mathcal{C} \right),$$

either side existing if the other does.

More generally, weighted homotopy limits in Kan-enriched categories can be expressed as conical homotopy limits by means of Lurie's "unstraightening" construction (which is an ∞-categorical generalisation of the classical "category of elements" construction).

A version of this result was proved in Martina Rovelli's recently published paper on '*Weighted limits in an* $(\infty, 1)$*-category*' (Applied Categorical Structures, 2021).

In this talk, I will give a much simpler, direct proof of a slightly stronger result.

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2. Straightening
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Weighted limits in simplicially-enriched categories

For the remainder of this talk, we will work in the setting of simplicially-enriched category theory.

Definition (weighted limits)

Let $W \colon \mathcal{A} \longrightarrow \mathbf{sSet}$ and $F \colon \mathcal{A} \longrightarrow \mathcal{C}$ be simplicial functors. A *limit of F weighted by W* is an object $\{W, F\} \in \mathcal{C}$, together with a simplicial natural transformation $W \longrightarrow \mathcal{C}(\{W, F\}, F-)$, such that the induced map

$$\mathcal{C}(C, \{W, F\}) \xrightarrow{\cong} \mathbf{Nat}(W, \mathcal{C}(C, F-))$$

is an isomorphism of simplicial sets for every object $C \in \mathcal{C}$.

Now suppose $\mathcal{C}$ is a Kan-enriched category. We can adapt the above definition to give a notion of "weighted homotopy limit" in $\mathcal{C}$ by:

1. replacing the simplicial natural transformations by "pseudo-natural transformations", and
2. replacing the defining isomorphism by an equivalence of Kan complexes.

To achieve step (1), we use the notion of "flexible weight".

The projective model structure

Let $\mathcal{A}$ be a small simplicial category.

Recall (projective model structure)

The simplicial functor category $\mathbf{Fun}(\mathcal{A}, \mathbf{sSet})$ admits a model structure – called the *projective model structure* – in which a simplicial natural transformation $\theta \colon F \longrightarrow G$ is:

- a *weak equivalence* if $\theta_A \colon FA \longrightarrow GA$ is a weak homotopy equivalence for every $A \in \mathcal{A}$;
- a *(trivial) fibration* if $\theta_A \colon FA \longrightarrow GA$ is a (trivial) Kan fibration for every $A \in \mathcal{A}$;
- a *cofibration* if it has the LLP wrt every projective trivial fibration.

In particular, a simplicial functor $F \colon \mathcal{A} \longrightarrow \mathbf{sSet}$ is a fibrant object in the projective model structure iff it takes values in Kan complexes.

Flexible weights

Definition (flexible weight)

A simplicial functor $W\colon \mathcal{A} \longrightarrow \mathbf{sSet}$ is *flexible* if it is a cofibrant object in the projective model structure on $\mathbf{Fun}(\mathcal{A}, \mathbf{sSet})$, i.e. if it has the LLP wrt every projective trivial fibration.

Definition (flexible replacement)

A *flexible replacement* of a weight $W\colon \mathcal{A} \longrightarrow \mathbf{sSet}$ is a flexible weight $W^c\colon \mathcal{A} \longrightarrow \mathbf{sSet}$ together with a weak equivalence $W^c \xrightarrow{\sim} W$.

The projective model structure on $\mathbf{Fun}(\mathcal{A}, \mathbf{sSet})$ is compatible with its standard simplicial enrichment. In particular, we have:

Proposition

Suppose that $W\colon \mathcal{A} \longrightarrow \mathbf{sSet}$ is a flexible weight and that $F\colon \mathcal{A} \longrightarrow \mathbf{sSet}$ is valued in Kan complexes. Then the simplicial set $\mathbf{Nat}(W, F)$ of simplicial natural transformations from W to F is a Kan complex.

Weighted homotopy limits

Let $\mathcal{A}$ be a small simplicial category and $\mathcal{C}$ a Kan-enriched category.

Definition (weighted homotopy limit)

Let $W\colon \mathcal{A} \longrightarrow \mathbf{sSet}$ and $F\colon \mathcal{A} \longrightarrow \mathcal{C}$ be simplicial functors. A *homotopy limit of F weighted by W* is an object $\{W, F\}_{ho}$ of $\mathcal{C}$, together with a simplicial natural transformation $W^c \longrightarrow \mathcal{C}(\{W, F\}_{ho}, F-)$ from a flexible replacement of W, such that the induced map

$$\mathcal{C}(C, \{W, F\}_{ho}) \xrightarrow{\sim} \mathbf{Nat}(W^c, \mathcal{C}(C, F-))$$

is an equivalence of Kan complexes for every object $C \in \mathcal{C}$.

Definition (conical homotopy limit)

Let $F\colon \mathcal{A} \longrightarrow \mathcal{C}$ be a simplicial functor. A *conical homotopy limit* of F is a homotopy limit of F weighted by the terminal weight $\Delta 1\colon \mathcal{A} \longrightarrow \mathbf{sSet}$.

We will denote a conical homotopy limit of F by $\mathrm{holim} F$. By definition, it has the homotopical universal property:

$$\mathcal{C}(C, \mathrm{holim} F) \simeq \mathbf{Nat}((\Delta 1)^c, \mathcal{C}(C, F-)).$$

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The straightening theorem

Recall (homotopy coherent realisation)

The homotopy coherent realisation functor $\mathfrak{C}\colon \mathbf{sSet} \longrightarrow \mathbf{sSet\text{-}Cat}$ is the cocontinuous extension of the functor $\mathfrak{C}\colon \Delta \longrightarrow \mathbf{sSet\text{-}Cat}$ that sends $[n]$ to the "homotopy coherent n-simplex". It is left adjoint to the homotopy coherent nerve functor.

Theorem (Lurie)

For every simplicial set B, the straightening–unstraightening adjunction

$$\begin{tikzcd} \mathbf{Fun}(\mathfrak{C}(B), \mathbf{sSet}) \arrow[r, bend right, "Un_B"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet}/B \arrow[l, bend right, "St_B"'] \end{tikzcd}$$

is a Quillen equivalence between the projective model structure on $\mathbf{Fun}(\mathfrak{C}(B), \mathbf{sSet})$ and the covariant model structure on $\mathbf{sSet}/B$.

Corollary

For every simplicial functor $W\colon \mathfrak{C}(B) \longrightarrow \mathbf{Kan}$, the counit morphism

$St_B(Un_B(W)) \xrightarrow{\sim} W$ witnesses the weight $St_B(Un_B(W))$ as a flexible replacement of W.

Straightening via left Kan extension

We can describe the straightening construction in the following two steps.

Observation (the fundamental weight)

Let X be a simplicial set. The straightening of the the terminal object $1_X \colon X \longrightarrow X$ of $\mathbf{sSet}/X$ is the composite simplicial functor

$$\mathfrak{C}(X) \longrightarrow \mathfrak{C}(X^{\triangleleft}) \xrightarrow{\mathrm{Hom}_{\mathfrak{C}(X^{\triangleleft})}(\perp,-)} \mathbf{sSet},$$

which sends an object $x \in \mathfrak{C}(X)$ to the simplicial hom-set $\mathrm{Hom}_{\mathfrak{C}(X^{\triangleleft})}(\perp, x)$. We call it the *fundamental weight* of X and denote it by $L_X \colon \mathfrak{C}(X) \longrightarrow \mathbf{sSet}$.

Proposition

*Let $p \colon X \longrightarrow B$ be a morphism of simplicial sets. Its straightening $St_B(X,p) \colon \mathfrak{C}(B) \longrightarrow$ **sSet** is the simplicial left Kan extension of the fundamental weight $L_X \colon \mathfrak{C}(X) \longrightarrow$ **sSet** along the simplicial functor $\mathfrak{C}(p) \colon \mathfrak{C}(X) \longrightarrow \mathfrak{C}(B)$.*

$$\begin{tikzcd} \mathfrak{C}(X) \arrow[rr, "\mathfrak{C}(p)"] \arrow[dr, "L_X"'] & \arrow[d, phantom, "\overset{Lan}{\Longrightarrow}"] & \mathfrak{C}(B) \arrow[dl, "St_B(X,p)"] \\ & \mathbf{sSet} & \end{tikzcd}$$

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Conical homotopy limits and straightening

Recall the corollary to the straightening theorem.

Corollary

For every simplicial functor $W \colon \mathfrak{C}(B) \longrightarrow \mathbf{Kan}$, the counit morphism $St_B(Un_B(W)) \longrightarrow W$ witnesses the weight $St_B(Un_B(W))$ as a flexible replacement of W.

In particular, taking W to be the terminal weight yields:

Proposition

For every simplicial set X, the fundamental weight $L_X \colon \mathfrak{C}(X) \longrightarrow \mathbf{sSet}$ is a flexible replacement of the terminal weight $\Delta 1 \colon \mathfrak{C}(X) \longrightarrow \mathbf{sSet}$.

Hence, by the definition of conical homotopy limits, we have:

Proposition

Let X be a simplicial set, $\mathcal{C}$ a Kan-enriched category, and $F \colon \mathfrak{C}(X) \longrightarrow \mathcal{C}$ a simplicial functor. Then we have

$$\{L_X, F\}_{ho} \simeq \mathrm{holim} F,$$

either side existing if the other does.

Weighted homotopy limits and left Kan extensions

We will use the following homotopical version of a standard result, which expresses the cocontinuity of weighted limits in the weight.

Proposition

Let $K \colon \mathcal{A} \longrightarrow \mathcal{B}$ be a simplicial functor between small simplicial categories, and let $W \colon \mathcal{A} \longrightarrow \mathbf{sSet}$ be a flexible weight. Then for any Kan-enriched category $\mathcal{C}$ and simplicial functor $F \colon \mathcal{B} \longrightarrow \mathcal{C}$, we have

$$\{\mathrm{Lan}_K W, F\}_{ho} \simeq \{W, FK\}_{ho},$$

either side existing if the other does.

Proof.

The functor $\mathrm{Lan}_K \colon \mathbf{Fun}(\mathcal{A}, \mathbf{sSet}) \longrightarrow \mathbf{Fun}(\mathcal{B}, \mathbf{sSet})$ preserves flexible weights because its right adjoint $\mathrm{res}_K \colon \mathbf{Fun}(\mathcal{B}, \mathbf{sSet}) \longrightarrow \mathbf{Fun}(\mathcal{A}, \mathbf{sSet})$ preserves projective trivial fibrations. By the universal property of simplicial left Kan extensions, we have the natural isomorphism $\mathbf{Nat}(\mathrm{Lan}_K W, \mathcal{C}(C, F-)) \cong \mathbf{Nat}(W, \mathcal{C}(C, FK-))$. Hence, if the weighted homotopy limit $\{\mathrm{Lan}_K W, F\}_{ho}$ exists, it also has the universal property of the weighted homotopy limit $\{W, FK\}_{ho}$, and vice versa. $\square$

Weighted homotopy limits as conical homotopy limits

Theorem

Let B be a simplicial set, $W\colon \mathfrak{C}(B) \longrightarrow \textbf{sSet}$ a weight, and let $p\colon Un_B(W^f) \longrightarrow B$ denote the unstraightening of a projective fibrant replacement of W. Also, let $\mathcal{C}$ be a Kan-enriched category and $F\colon \mathfrak{C}(B) \longrightarrow \mathcal{C}$ a simplicial functor. Then we have

$$\{W, F\}_{ho} \simeq \operatorname{holim}\left(\mathfrak{C}(Un_B(W^f)) \xrightarrow{\mathfrak{C}(p)} \mathfrak{C}(B) \xrightarrow{F} \mathcal{C} \right),$$

either side existing if the other does.

Proof.

$$\{W, F\}_{ho} \simeq \{St_B Un_B(W^f), F\}_{ho} \simeq \{L_{Un_B(W^f)}, F \circ \mathfrak{C}(p)\}_{ho} \simeq \operatorname{holim}(F \circ \mathfrak{C}(p)) \qquad \square$$

Remark

More generally, let B be a simplicial set, $\mathcal{B}$ a simplicial category, and $\phi\colon \mathfrak{C}(B) \longrightarrow \mathcal{B}$ a DK-equivalence. Then one can express homotopy limits weighted by simplicial functors $\mathcal{B} \longrightarrow \textbf{sSet}$ in terms of the unstraightening construction Un_ϕ relative to ϕ.